ECON 1110: Intermediate Macro

Lecture 7 + 8 Supplement

1 Government Budgeting

At any given time, let the government surplus be S_t. We will assume that the surplus is some fraction a of GDP, so that

$$S_t = aY_t$$

Note that a can be either positive (a surplus) or negative (a deficit). The evolution of the debt level will thus be

$$D_t = (1+r)D_{t-1} - S_t$$

Now we want to find the evolution of the normalized debt to GDP ratio $d_t = D_t/Y_t$. To do this we divide by Y_t

$$\begin{aligned}
&\frac{D_t}{Y_t} = (1+r)\frac{D_{t-1}}{Y_t} - \frac{S_t}{Y_t} \\
\Rightarrow\ &\frac{D_t}{Y_t} = (1+r)\frac{D_{t-1}}{Y_t}\frac{Y_{t-1}}{Y_{t-1}} - \frac{S_t}{Y_t} \\
\Rightarrow\ &\frac{D_t}{Y_t} = (1+r)\frac{D_{t-1}}{Y_{t-1}}\frac{Y_{t-1}}{Y_t} - \frac{S_t}{Y_t} \\
\Rightarrow\ &d_t = (1+r)d_{t-1}\frac{1}{1+g} - a \\
\Rightarrow\ &d_t = \left(\frac{1+r}{1+g}\right)d_{t-1} - a
\end{aligned}$$

To find the steady state value, we impose $d_t = d_{t-1} = d^*$ this yields

$$\begin{aligned} d^* &= \left(\frac{1+r}{1+g}\right) d^* - a \\ \Rightarrow \left[1 - \left(\frac{1+r}{1+g}\right)\right] d^* &= -a \\ \Rightarrow d^* &= \frac{-a}{1 - \left(\frac{1+r}{1+g}\right)} \\ \Rightarrow d^* &= \frac{(-a)(1+g)}{g-r} \end{aligned}$$

For the steady state to exist, we then need $r < g$.

1.1 Social Security

When thinking about social security, we have population growth at rate n so that

$$N' = (1+n)N$$

The young are taxed at rate t and the old receive benefits b. As such, a balanced government budget requires

$$\begin{aligned} bN &= tN' = t(1+n)N \\ \Rightarrow b &= (1+n)t \end{aligned}$$

Upon introduction of this policy, the old are clearly better off. The change in the present value of wealth for the young is

$$\Delta we = -t + \frac{b}{1+r} = -t + \frac{(1+n)t}{1+r} = t\left[\frac{1+n}{1+r} - 1\right] = \frac{t(n-r)}{1+r}$$

Thus they are better off only if $r \leq n$ and they are worse off otherwise.

2 Savings and Investment

The profits of the firm are

$$\pi = zF(K,N) - wN - I$$
$$\pi' = z'F(K',N') - w'N' + (1-d)K'$$

and the present value of their income is

$$V = \pi + \frac{\pi'}{1+r}$$

and capital evolves according to

$$K' = (1-d)K + I$$

Find the optimal choice for investment then amounts to

$$\frac{\partial V}{\partial I} = -1 + \frac{z'F_K(K',N') + (1-d)}{1+r} = 0$$
$$\Rightarrow z'F_K(K',N') + (1-d) = 1+r$$
$$\Rightarrow MPK' + (1-d) = 1+r$$

The optimality condition for labor utilization is

$$zF_N(K,N) = w \quad \Rightarrow \quad MPL = w$$
$$z'F_N(K',N') = w' \quad \Rightarrow \quad MPL' = w'$$

As we've derived previously, the optimality conditions for the consumer are

$$\begin{aligned} MRS_{c,c'} &= 1 + r \\ MRS_{\ell,c} &= w \\ MRS_{\ell',c'} &= w' \end{aligned}$$

Combining these with the firm conditions yields

$$\begin{aligned} MRS_{c,c'} &= MPK' + (1 - d) \\ MRS_{\ell,c} &= MPL \\ MRS_{\ell',c'} &= MPL' \end{aligned}$$

2.1 Standard Case

Let's use our usual utility function

$$\begin{aligned} U(c, c') &= u(c, \ell) + \beta u(c', \ell') \\ \text{where} \quad u(c, \ell) &= \log(c) + \eta \log(\ell) \end{aligned}$$

and production function

$$\begin{aligned} zF(K, N) &= zK^{\alpha} N^{1-\alpha} \\ z'F(K', N') &= z'(K')^{\alpha} (N')^{1-\alpha} \end{aligned}$$

Keep in mind that $\ell + N = 1$. Our equilibrium conditions are then

$$\begin{aligned} \frac{c'}{\beta c} &= \alpha z' \left(\frac{N'}{K'} \right)^{1-\alpha} + (1 - d) \\ \frac{\eta c}{1 - N} &= (1 - \alpha) z \left(\frac{K}{N} \right)^{\alpha} \\ \frac{\eta c'}{1 - N'} &= (1 - \alpha) z' \left(\frac{K'}{N'} \right)^{\alpha} \end{aligned}$$

2.2 Full Depreciation

Suppose that capital fully depreciates each period so that $d = 1$ and $K' = I$. Note that $Y = c + I$ and $Y' = c'$. Then we have

$$\begin{aligned}
&\frac{c'}{\beta c} = \alpha z' \left(\frac{N'}{K'}\right)^{1-\alpha} \\
\Rightarrow\ &\frac{c'}{\beta c} = \alpha \frac{Y'}{K'} \\
\Rightarrow\ &\frac{1}{\beta(Y-I)} = \frac{\alpha}{I} \\
\Rightarrow\ &\frac{I}{Y-I} = \alpha\beta \\
\Rightarrow\ &\frac{I}{Y} = \frac{\alpha\beta}{1+\alpha\beta}
\end{aligned}$$

This is the savings rate we had assumed as exogenous in the Solow model. Additionally we can show

$$\frac{c}{Y} = \frac{1}{\alpha\beta + 1}$$

Finally, from the labor market condition we can see

$$\begin{aligned}
&\frac{\eta c}{1-N} = (1-\alpha) z \left(\frac{K}{N}\right)^{\alpha} \\
\Rightarrow\ &\frac{\eta c}{1-N} = (1-\alpha)\frac{Y}{N} \\
\Rightarrow\ &\frac{\eta N}{1-N} = (1-\alpha)\frac{Y}{c} \\
\Rightarrow\ &\frac{\eta N}{1-N} = (1-\alpha)(\alpha\beta+1) \\
\Rightarrow\ &N = \frac{(1-\alpha)(\alpha\beta+1)}{\eta + (1-\alpha)(\alpha\beta+1)} \\
\Rightarrow\ &N = \frac{1-\alpha}{\frac{\eta}{\alpha\beta+1} + (1-\alpha)}
\end{aligned}$$

2.3 Extra Problem

Suppose now that there is not labor, only capital, so that

$$zF(K) = zK^\alpha \quad \text{and} \quad z'F(K') = z'(K')^\alpha$$

Further, we have a new utility function

$$u(c) = 1 - \exp(-c)$$

The implied marginal utility here is

$$u'(c) = \exp(-c)$$

and the consumer optimality condition is

$$\begin{aligned} MRS_{c,c'} &= 1 + r \\ \Rightarrow \ \exp(c' - c) &= 1 + r \end{aligned}$$

Combining this with the firm, we find

$$\begin{aligned} \exp(c' - c) &= \alpha z'(K')^{\alpha-1} \\ \Rightarrow \ \exp(z'I^\alpha + I - zK^\alpha) &= \frac{\alpha z'}{I^{1-\alpha}} \end{aligned}$$

Since the left-hand side is increasing and the right-hand side is decreasing, it is clear that this has a unique solution for I. To make things tractable, let's also assume that $\alpha = 1$ so that

$$
\begin{aligned}
&\exp((1+z')I - zK) = z' \\
\Rightarrow\ &(1+z')I - zK = \log(z') \\
\Rightarrow\ &(1+z')I = zK + \log(z') \\
\Rightarrow\ &K' = I = \frac{zK + \log(z')}{1+z'}
\end{aligned}
$$