

ECON 1110: Intermediate Macro

Lectures 4 + 5 Supplement

1 Malthusian Model

There is a fixed amount of land L . At any given time this is a population N this generation N' in the next generation. The production technology is given by

$$Y = zL^\alpha N^{1-\alpha}$$

where Y is output and $\alpha \in (0, 1)$ is a parameter. Since there is no investment, consumption is equal to output, $C = Y$. The rate of population growth is assumed to be an increasing function (g) of the consumption per person (C/N). This means

$$\frac{N'}{N} = g\left(\frac{C}{N}\right)$$

It is useful to define normalized quantities $y = Y/N$, $c = C/N$, and $\ell = L/N$. Additionally, the population growth rate n satisfies $1 + n = N'/N$. With this we can write

$$1 + n = g(c)$$

and using the production function

$$y = \frac{Y}{N} = \frac{zL^\alpha N^{1-\alpha}}{N} = zL^\alpha N^{-\alpha} = z\left(\frac{L}{N}\right)^\alpha = z\ell^\alpha$$

Thus we have

$$c = y = z\ell^\alpha$$

In equilibrium, we will have a constant population, meaning $n = 0$. Thus the standard of living c^* will satisfy

$$1 = g(c^*)$$

which uniquely determines c^* since g is increasing. From here we can find ℓ^* using

$$c^* = z(\ell^*)^\alpha \quad \Rightarrow \quad \ell^* = \left(\frac{c^*}{z}\right)^{1/\alpha}$$

Finally we can find the level of population with

$$\ell^* = \frac{L}{N^*} \quad \Rightarrow \quad N^* = \frac{L}{\ell^*}$$

2 Solow Model

Now we are dealing with capital (K) and labor (N). The production function takes the form

$$Y = zK^\alpha N^{1-\alpha}$$

We assume that population grows exogenous at rate n , so that $N' = (1 + n)N$. Output either goes to consumption or investment so that

$$Y = C + I$$

We assume that a fraction of output s is invested meaning

$$I = sY \quad \text{and} \quad C = (1 - s)Y$$

Capital evolves due to both investment and depreciation (d) according to

$$\begin{aligned} K' &= I + (1 - d)K \\ &= sY + (1 - d)K \\ &= szK^\alpha N^{1-\alpha} + (1 - d)K \end{aligned}$$

Instead of keeping track of K and N , it is easier to instead track their ratio $k = K/N$. To do this we divide the above by N

$$\begin{aligned} \frac{K'}{N} &= \frac{szK^\alpha N^{1-\alpha}}{N} + \frac{(1 - d)K}{N} \\ \Rightarrow \frac{K'}{N} \cdot \frac{N'}{N'} &= sz \left(\frac{K}{N} \right)^\alpha + (1 - d) \left(\frac{K}{N} \right) \\ \Rightarrow (1 + n)k' &= szk^\alpha + (1 - d)k \\ \Rightarrow k' &= \frac{szk^\alpha + (1 - d)k}{1 + n} \end{aligned}$$

Great, now we just have to deal with the capital to labor ratio k . To find the steady state value, we simply equate k and k' and solve

$$\begin{aligned} k &= \frac{szk^\alpha + (1 - d)k}{1 + n} \\ \Rightarrow (1 + n)k &= szk^\alpha + (1 - d)k \\ \Rightarrow (n + d)k &= szk^\alpha \\ \Rightarrow k^{1-\alpha} &= \frac{sz}{n + d} \\ \Rightarrow k^* &= \left(\frac{sz}{n + d} \right)^{\frac{1}{1-\alpha}} \end{aligned}$$

Using this value, we can then find output and consumption

$$y^* = z(k^*)^\alpha \quad \text{and} \quad c^* = (1 - s)y^*$$

3 Endogenous Growth

Now we'll be looking at the growth of human capital and how it leads to economic growth. At any given time, let the human capital of the current generation be H and that of the next generation be H' . Agents spend some fraction of their time u working and the remainder $1 - u$ studying. Thus human capital evolves according to

$$H' = b(1 - u)H$$

where $b > 0$ measures the overall efficiency of education. With this, we the growth of human capital to be

$$\frac{H' - H}{H} = \frac{b(1 - u)H - H}{H} = b(1 - u) - 1$$

Firms employ workers and pay them a wage w per unit of utilized human capital, meaning the total wage bill is

$$wuH$$

Workers have a productivity z per unit of utilized human capital, so that output is $Y = zuH$. We can then write firm profit as

$$\pi = zuH - wuH$$

From this we can infer that the wage is simply the productivity, $w = z$, otherwise the firm would hire either zero or infinitely many workers, which we want to avoid. The growth rate of output is

$$\frac{Y' - Y}{Y} = \frac{zuH' - zuH}{zuH} = \frac{H' - H}{H} = b(1 - u) - 1$$

So output will grow at this rate in perpetuity. If this number is positive ($b(1 - u) > 1$), as we will usually assume, output will be increasing without bound. In the case that it is

negative ($b(1 - u) < 1$), output will converge to zero.