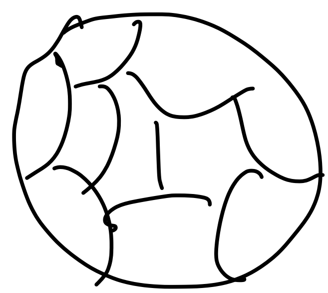
Malthus

① $Y = zK^{\alpha}L^{1-\alpha}$

$$\frac{Y}{L} = zK^{\alpha}L^{-\alpha}$$

$$\frac{Y}{L} = z\left(\frac{K}{L}\right)^{\alpha}$$

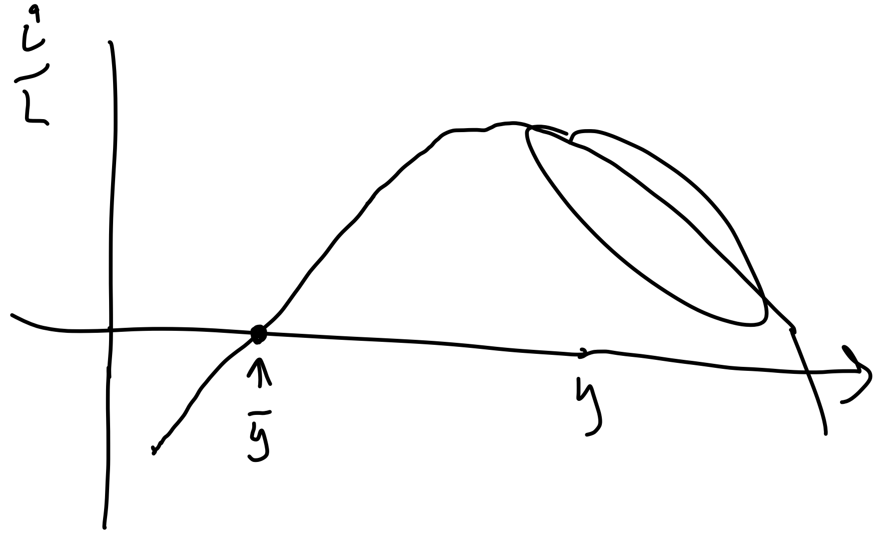
$$y = zk^{\alpha}$$



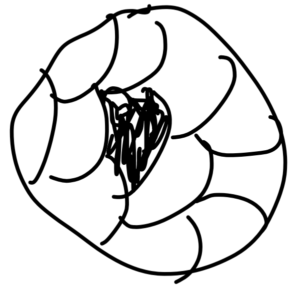
$\dot{L} \equiv \frac{dL}{dt}$

② $\frac{\dot{L}}{L} = \theta(y - \bar{y})$

$y \to \bar{y} \to \frac{\dot{L}}{L} = 0 \to L$ const.

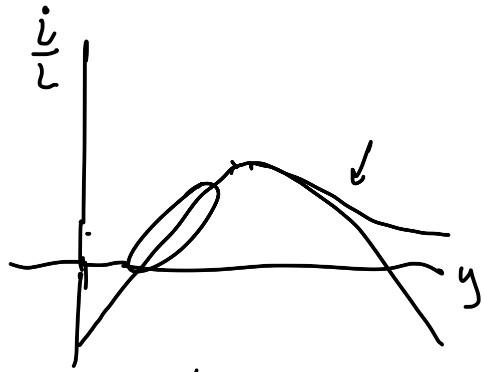


$Y = zK^{\alpha}L^{1-\alpha}$



$\frac{\dot{i}}{L} = \theta(y - y_0)$

$\frac{Y}{L} = z\left(\frac{K}{L}\right)^{\alpha}$



$y = zk^{\alpha} \leftarrow$

$y = f(k)$

Leontieff

$Y = \min\{aK, bL\}$

$\frac{Y}{L} = \min\{a\frac{K}{L}, b\} \rightarrow y = \min\{ak, b\}$

Growth rate tricks

$y(t)$ $\dot{y}$ α

y_t, y_{t+1}

$\downarrow$

$y(t), y(t+\Delta)$

$\rightarrow \lim_{\Delta \to 0} \frac{y(t+\Delta) - y(t)}{\Delta} = \dot{y}$

$\beta \in (0,1)$

$\rho > 0 \rightarrow \exp(-\rho t)$

$\dot{y}$ derivative

$\frac{\dot{y}}{y}$ growth rate $\rightarrow \frac{\dot{y}}{y} = g \rightarrow \dot{y} = gy$

$\rightarrow \frac{dy}{dt} = gy \rightarrow \frac{dy}{y} = g\,dt \rightarrow \log(y) = gt + C$

$\rightarrow y(0) = 1 \rightarrow C = 0 \rightarrow \log(y) = gt \rightarrow y(t) = \exp(gt)$

$\left\{ \frac{y(1) - y(0)}{y(0)} = \frac{\exp(g)-1}{1} = \exp(g) - 1 \right.$

$y(t) \quad x(t)$

$\downarrow \quad \downarrow$

$\frac{\dot{y}}{y} = g_y \quad \frac{\dot{x}}{x} = g_x$

$\frac{dy}{dt}$

$\downarrow$

$\rightarrow \frac{\dot{y}}{y} = \frac{d\log(y)}{dt}$

$z(t) = x(t) \cdot y(t)$

$g_z = \frac{\dot{z}}{z} = \frac{x\dot{y} + \dot{x}y}{xy} = \frac{\dot{y}}{y} + \frac{\dot{x}}{x} = g_y + g_x$

$g_z = \frac{d\log(z)}{dt} = \frac{d}{dt}\left[\log(z)\right]$

$= \frac{d}{dt}\left[\log(x \cdot y)\right] = \frac{d}{dt}\left[\log(x) + \log(y)\right]$

$= \frac{d}{dt}\log(x) + \frac{d}{dt}\log(y)$

$= g_x + g_y$

Quotient

$z = x/y \rightarrow g_z = g_x - g_y$

Power

$z = x^\alpha \rightarrow \log(z) = \alpha\log(x) \rightarrow g_z = \alpha g_x$

$\alpha = -1 \rightarrow g_z = -g_x$

Cobb-Douglas

$Y = zK^\alpha L^{1-\alpha}$

$g_Y = g_z + g(K^\alpha) + g(L^{1-\alpha}) \qquad y = \frac{Y}{L} \quad k = \frac{K}{L}$

$g_Y = g_z + \alpha g_K + (1-\alpha) g_L \qquad y = zk^\alpha$

$k = \frac{K}{L}$

$\rightarrow g_Y - g_L = g_z + \alpha g_K - \alpha g_L$

$g_Y - g_L = g_z + \alpha(g_K - g_L)$

$g_k = 0$

$\downarrow$

$g_K = g_L$

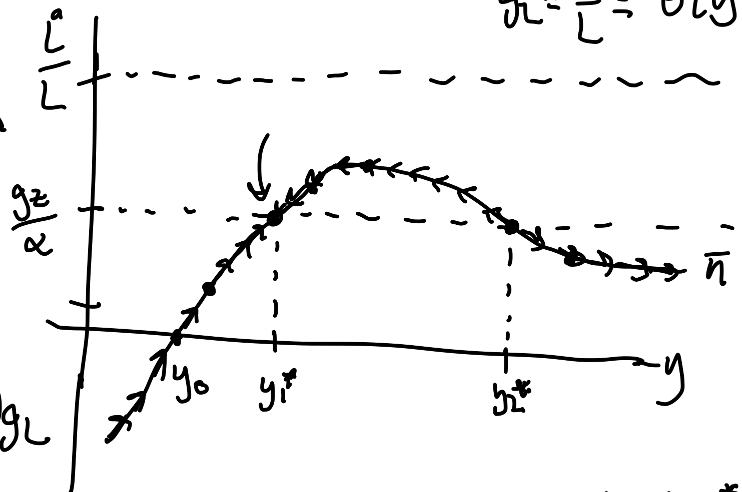
$\rightarrow g_y = g_z + \alpha g_k$

$y = \frac{Y}{L}$ $\quad g_Y$ known

$g_y = g_Y - g_L = g_Y - n$

$g_L = \frac{\dot{L}}{L} = \theta(y - y_0)$

<u>Malthus</u>



$\boxed{\begin{matrix} z(0) \\ L(0) \end{matrix}} \rightarrow y(0)$

$Y = zK^\alpha L^{1-\alpha}$

$\rightarrow y = zk^\alpha$

$g_Y = g_z + \alpha g_K + (1-\alpha) g_L$

$g_Y = g_z + (1-\alpha) g_L$

$\searrow g_Y - g_L = g_z - \alpha g_L$

$\rightarrow \boxed{g_y = g_z - \alpha g_L}$

$g_y > 0?$ $\quad g_L < \frac{g_z}{\alpha}$

$y \rightarrow y^*$ const.

$g_y = 0 \rightarrow g_z = \alpha g_L$

$\rightarrow g_z = \alpha\theta(y^* - y_0)$

$\rightarrow y^* = y_0 + \frac{g_z}{\alpha\theta}$

$M = \frac{\partial Y}{\partial K} = z\alpha K^{\alpha-1} L^{1-\alpha}$

$= z\alpha\left(\frac{K}{L}\right)^{\alpha-1}$

$f(k) \equiv F(k,1,A)$

$\rightarrow f'(k) = F_K(k,1,A)$

$g_M = g_z + (1-\alpha) g_L = (g_z - \alpha g_L) + g_L$

$= g_y + g_L$

$\lambda^{m-1} = \lambda^0 = 1$

$\lambda = \frac{1}{L}$

<u>Factor Prices</u>

$K \rightarrow R$

$L \rightarrow w$

$R = F_K(K, L, A) = F_K\left(\frac{K}{L}, 1, A\right)$

$= F_K(k, 1, A) = f'(k)$

$w = F_L(K, L, A) = F_L\left(\frac{K}{L}, 1, A\right) = F_L(k, 1, A)$

Homogeneity (m) $m=1$

$\rightarrow F(K,L,A) = F_K(K,L,A)\cdot K + F_L(K,L,A)\cdot L$

$\rightarrow F(\lambda K, \lambda L, A) = \lambda^m F(K,L,A) \quad \forall \lambda, K, L, A$

$\frac{\partial}{\partial \lambda} \rightarrow F_K(\lambda K, \lambda L, A)\cdot K + F_L(\lambda K, \lambda L, A)\cdot L = m\cdot \lambda^{m-1}\cdot F(K,L,A)$

$\frac{\partial}{\partial K} \rightarrow \lambda \cdot F_K(\lambda K, \lambda L, A) = \lambda^m F_K(K,L,A)$

$\rightarrow F_K(\lambda K, \lambda L, A) = \lambda^{m-1} F_K(K,L,A)$ Homog. $m-1$ F_K

$\frac{\partial}{\partial L} \rightarrow$ Homog. $m-1$ F_L

$w = F_L(K,L,A) = F_L\left(\frac{K}{L}, 1, A\right) = F_L(k,1,A)$

$F(K,L,A) = K\cdot F_K(K,L,A) + L\cdot F_L(K,L,A)$

$\hookrightarrow \frac{F(K,L,A)}{L} = \frac{K}{L}\cdot F_K(K,L,A) + F_L(K,L,A)$

$\rightarrow F\left(\frac{K}{L}, 1, A\right) = \frac{K}{L}\cdot F_K\left(\frac{K}{L}, 1, A\right) + F_L\left(\frac{K}{L}, 1, A\right)$ $\quad f(k,A)$

$\rightarrow F(k,1,A) = k\left(F_K(k,1,A)\right) + F_L(k,1,A)$

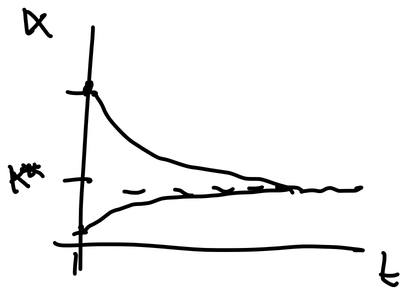
$\rightarrow f(k) = k\cdot f'(k) + w \qquad \boxed{\dot{k} = sf(k) - (\delta + n)k}$

$\rightarrow \begin{cases} w = f(k) - k f'(k) \\ R = f'(k) \end{cases}$

$\rightarrow w = y - k\cdot R$

$\rightarrow y = kR + w\cdot 1$

$\times L \quad Y = R\cdot K + wL$



$\rightarrow \dot{k} = sf(k) - (\delta+n)k \leftarrow f(k) = k^{\alpha}$

$\rightarrow k^* \quad \dot{k} = 0 \rightarrow \boxed{sf(k^*) = (\delta+n)k^*}$

$k(t) = \dfrac{K(t)}{L(t)} \rightarrow K(t) = k(t)\cdot L(t)$

long-run $\rightarrow$ $K(t) \approx k^* \cdot L(t)$

$K(t) \approx k^* \cdot L(0)\exp(nt)$

$g_K = g_k + g_L = g_k + n$

Golden Rule $\quad \dot{k} = sf(k) - (\delta+n)k$

$\rightarrow sf(k^*) = (\delta+n)k^*$

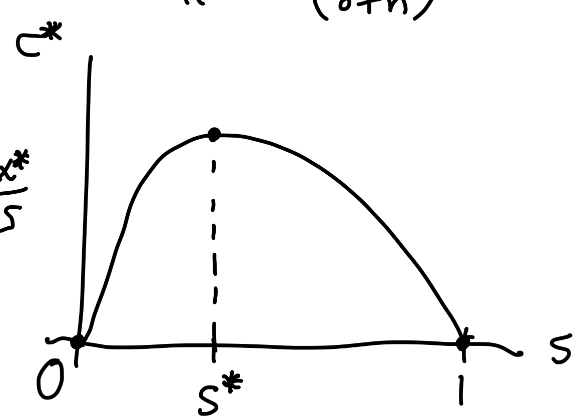
C-D: $sk^{\alpha} = (\delta+n)k$

$k^* = \left(\dfrac{s}{\delta+n}\right)^{1/(1-\alpha)}$

$y^* = f(k^*)$

$c^*(s) = (1-s)\cdot f(k^*(s))$

$c'(s) = (1-s)\cdot f'(k^*(s))\cdot \dfrac{\partial k^*}{\partial s}$
$\quad - f(k^*(s))$



$c^*(s) = (1-s) f(k^*(s))$

$= f(k^*(s)) - sf(k^*(s))$

$= f(k^*(s)) - (\delta+n)k^*(s)$

$\dfrac{\partial c^*(s)}{\partial s} = \left[f'(k^*(s)) - (\delta+n)\right]\cdot \dfrac{\partial k^*}{\partial s} = 0$

$\boxed{f'(k^*(s)) = \delta+n}$

C-D:

$f(k) = k^{\alpha}$

$f'(k) = \alpha k^{\alpha-1}$

$k^* = \left(\dfrac{s}{\delta+n}\right)^{1/(1-\alpha)}$

$f'(k^*) = \alpha\cdot\dfrac{\delta+n}{s}$

$\alpha\cdot\dfrac{\cancel{\delta+n}}{s} = \cancel{\delta+n}$

$\rightarrow \boxed{s = \alpha}$

Solow outcome

$k = \frac{K}{L}$ $\quad y = \frac{Y}{L}$ $\quad \rightarrow \quad y = f(k)$

$\dot{k} = sf(k) - (\delta + n)k$

$sf(k^*) = (\delta + n)k^*$

$\rightarrow (s, S)$

Cobb-Douglas $\quad Y = zK^{\alpha}L^{1-\alpha}$

$y = zk^{\alpha}$

$\dot{k} = szk^{\alpha} - (\delta + n)k$

$k^* = \left(\frac{sz}{\delta + n}\right)^{1/(1-\alpha)}$ $\quad y^* = z\left(\frac{sz}{\delta + n}\right)^{\alpha/(1-\alpha)}$

$1 + \frac{\alpha}{1-\alpha}$ $\quad \rightarrow y^* = z^{1/(1-\alpha)}\left(\frac{s}{\delta + n}\right)^{\alpha/(1-\alpha)}$

$= \frac{1-\alpha}{1-\alpha} + \frac{\alpha}{1-\alpha}$

$= \frac{1}{1-\alpha}$

$\alpha = 1$ (AK)

$\dot{k} = szk - (\delta + n)k$

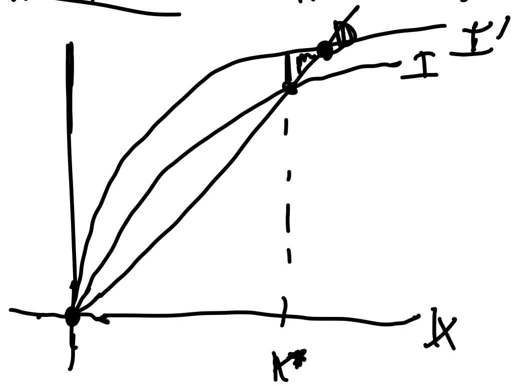
$\dot{k} = (sz - (\delta + n))k$

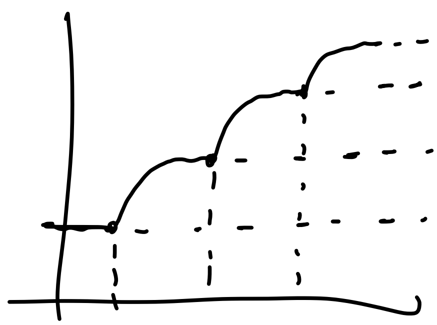
$\frac{\dot{k}}{k} = sz - (\delta + n)$

Transition

$\qquad\quad I \quad - \quad D$

$\dot{k} = sf(k) - (\delta + n)k = szk^{\alpha} - (\delta + n)k$





Continual

$\frac{\dot{z}}{z} = g$ $\quad k$

$\dot{K} = sY - \delta K$

$\frac{\dot{K}}{K} = s \cdot \frac{Y}{K} - \delta$

$g_Y = g_K$

$Y = zK^{\alpha}L^{1-\alpha} \rightarrow g_Y = g_z + \alpha g_K + (1-\alpha) g_L$

$\frac{Y}{L} = zK^{\alpha}L^{-\alpha} = z\left(\frac{K}{L}\right)^{\alpha}$ $\quad \downarrow$

$g_Y = g + \alpha g_K + (1-\alpha)n$

$\rightarrow y = zk^{\alpha} \leftarrow$

$g_K = g + \alpha g_K + (1-\alpha)n$

$z^{1/(1-\alpha)} \cdot L$

$(1-\alpha) g_K = g + (1-\alpha)n$

$\tilde{y} = \frac{y}{z^{1/(1-\alpha)}} = \frac{Y}{z^{1/(1-\alpha)} \cdot L}$ $\quad g_C = g_I = g_Y = g_K = \frac{g}{1-\alpha} + n$

$\tilde{k} = \frac{k}{z^{1/(1-\alpha)}} = \frac{K}{z^{1/(1-\alpha)} \cdot L}$

Renormalize

$$1+\frac{\alpha}{1-\alpha}=\frac{1}{1-\alpha}$$

$$y = zk^{\alpha}$$

$$\tilde{k} = \frac{k}{z^{1/(1-\alpha)}}$$

$$\tilde{y} = \frac{y}{z^{1/(1-\alpha)}}$$

$$\tilde{y} = \frac{zk^{\alpha}}{z^{1/(1-\alpha)}} = \frac{z\cdot\left(\tilde{k}\cdot z^{1/(1-\alpha)}\right)^{\alpha}}{z^{1/(1-\alpha)}} = \frac{z^{1/(1-\alpha)}\cdot\tilde{k}^{\alpha}}{z^{1/(1-\alpha)}} = \tilde{k}^{\alpha}$$

$$\tilde{y} = \tilde{k}^{\alpha}$$

$$\dot{k} = szk^{\alpha} - (\delta+n)k$$

$$\frac{\dot{k}}{k} = \frac{sz}{k^{1-\alpha}} - (\delta+n)$$

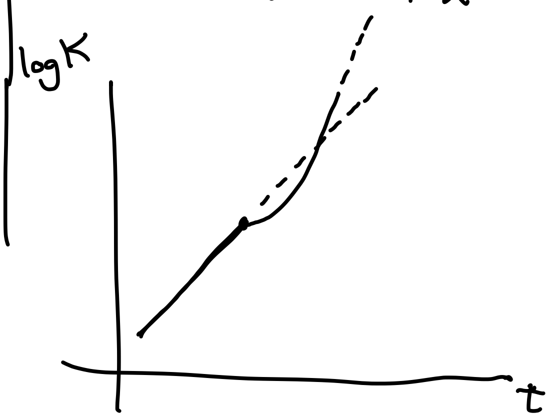
$$= \frac{s}{\tilde{k}^{1-\alpha}} - (\delta+n)$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{k}}{k} - \frac{g}{1-\alpha} = \frac{s}{\tilde{k}^{1-\alpha}} - (\delta+n) - \frac{g}{1-\alpha}$$

$$\rightarrow \boxed{\dot{\tilde{k}} = s\tilde{k}^{\alpha} - \left(\delta+n+\frac{g}{1-\alpha}\right)\tilde{k}}$$

Steady state

$$s\tilde{k}^{\alpha} = \left(\delta+n+\frac{g}{1-\alpha}\right)\tilde{k} \rightarrow \tilde{k} = \left(\frac{s}{\delta+n+\frac{g}{1-\alpha}}\right)^{1/(1-\alpha)}$$



$$\log K = \log\left(\tilde{k}\cdot L\cdot z^{1/(1-\alpha)}\right)$$

$$= \log\tilde{k} + \log L + \frac{1}{1-\alpha}\log z$$

$$g_K = g_{\tilde{k}} + n + \frac{g}{1-\alpha}$$

Technology

$$Y = \tilde{F}(A,K,L)$$

1. $Y = AF(K,L)$
2. $Y = F(AK,L)$
3. $Y = F(K,AL)$

$$Y = A_1F(A_2K, A_3L)$$

$$= A_1\cdot(A_2K)^{\alpha}(A_3L)^{1-\alpha}$$

$$= \left[A_1A_2^{\alpha}A_3^{1-\alpha}\right]\cdot K^{\alpha}L^{1-\alpha}$$

$$= \left(\left[A_1A_2^{\alpha}A_3^{1-\alpha}\right]^{1/\alpha}\cdot K\right)^{\alpha}L^{1-\alpha}$$

CES $\rightarrow$ $$Y = \left(\alpha K^{1/(\varepsilon-1)} + (1-\alpha)L^{1/(\varepsilon-1)}\right)^{\varepsilon-1}$$

Uzawa CES

$$Y = F(K, L, A) = \left[\alpha (AK)^{1-\rho} + (1-\alpha) L^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

$$\dot{K} = sF(K, L, A) - \delta K$$

$$g_K = \frac{\dot{K}}{K} = \frac{sF(K,L,A)}{K} - \delta = s \cdot \left[\alpha (A)^{1-\rho} + (1-\alpha)\left(\frac{L}{K}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}} - \delta$$

$$\rightarrow g_K = s\,\alpha^{\frac{1}{1-\rho}} \cdot A - \delta$$

$$Y = F(K, L, A) = \left[\alpha K^{1-\rho} + (1-\alpha)(AL)^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

$$g_K = \frac{\dot{K}}{K} = \frac{sF(K,L,A)}{K} - \delta = s \cdot \left[\alpha + (1-\alpha)\left(\frac{AL}{K}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}} - \delta$$

$$g_K = g_A + g_L = g + n$$

Solow + Technology

$$\frac{\dot{A}}{A} = g \qquad Y = F(K, AL) \rightarrow \frac{Y}{AL} = \frac{F(K, AL)}{AL} = F\left(\frac{K}{AL}, 1\right) = f\left(\frac{K}{AL}\right)$$

$$\frac{\dot{L}}{L} = n \qquad y = \frac{Y}{AL} \quad k = \frac{K}{AL} \qquad y = f(k)$$

$$\dot{K} = sY - \delta K$$

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{sY - \delta K}{K} - g - n$$

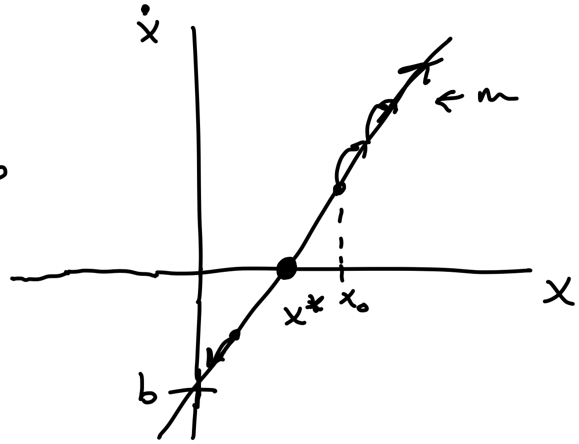
$$\frac{\dot{k}}{k} = s \cdot \frac{Y}{K} - (\delta + g + n) = s \cdot \frac{y}{k} - (\delta + g + n)$$

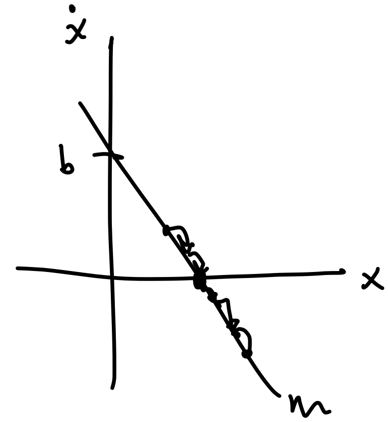
$$\rightarrow \dot{k} = sy - (\delta + g + n)k = sf(k) - (\delta + g + n)k$$

Diff. Eq.

$$\begin{cases} \dot{x}(t) = m \cdot x(t) + b \\ x(0) = x_0 \end{cases}$$

$x^* = -\frac{b}{m}$





$$x^g(t) = \underbrace{-\frac{b}{m}}_{x^*} + d \cdot \exp(mt)$$

$\rightarrow$ $d \cdot m \cdot \exp(mt) = -b + d \cdot m \cdot \exp(mt) + b$

$\dot{x}^g(t) = d \cdot m \cdot \exp(mt)$

$0 = 0$ ✓

$x(0) = x_0$

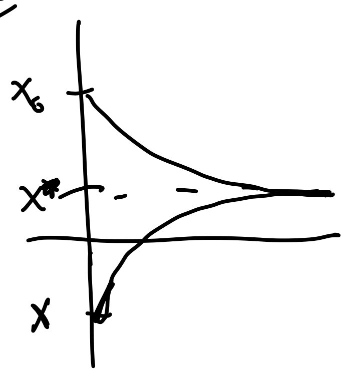
$x_0 = x^g(0) = -\frac{b}{m} + d$

$\rightarrow$ $d = x_0 + \frac{b}{m} = x_0 - x^*$

$x(t) = -\frac{b}{m} + \left(x_0 + \frac{b}{m}\right) \exp(mt)$

$x(t) = x^* + \left(x_0 - x^*\right) \exp(mt)$

$m < 0$



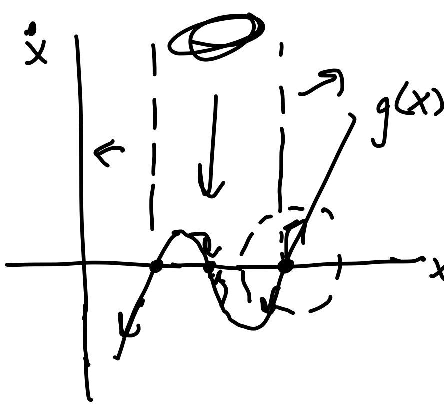
$m < 0 \rightarrow x(t) = x_0 \cdot \exp(mt) + x^* \cdot \left[1 - \exp(mt)\right]$

$m > 0 \rightarrow x(t) = x_0 + (x_0 - x^*)(\exp(mt) - 1)$

Non-linear

$\dot{x}(t) = g(x(t))$

$g(x^*) = 0$



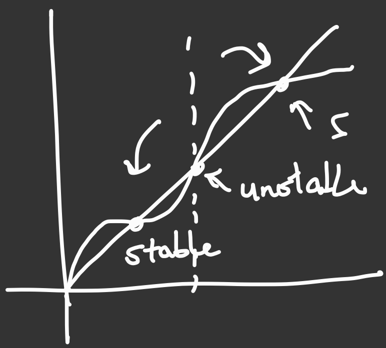
$\rightarrow m = g'(x^*)$

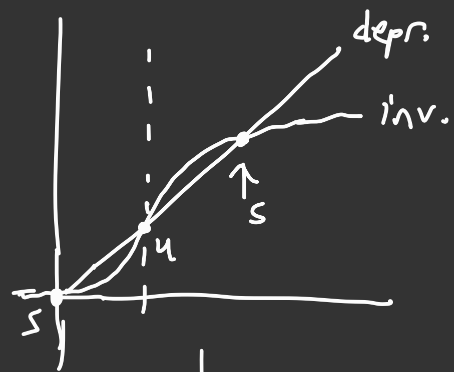
$b = -g'(x^*) \cdot x^*$

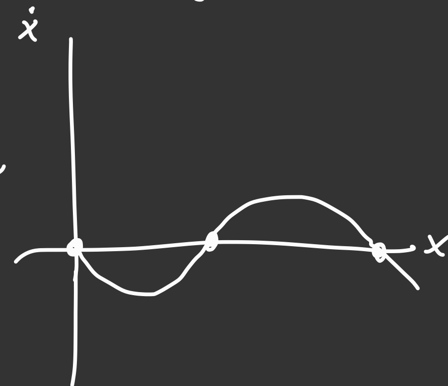
$\dot{x}(t) \approx g'(x^*)(x(t) - x^*)$

$g(x) \approx g(x^*) + (x - x^*) \cdot g'(x^*)$

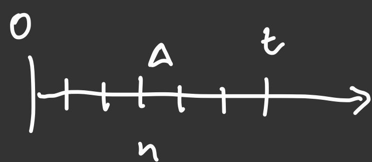
$g(x) = g'(x^*) \cdot (x - x^*)$







Discounting



$\Delta \cdot n = t$

$\rho > 0$

$$v(t|\Delta) = (1-\Delta\rho)^n = (1-\Delta\rho)^{t/\Delta}$$

$\beta \leftrightarrow \Delta\rho$

limit $\Delta \to 0$

$$\lim_{\Delta \to 0} (1-\Delta\rho)^{t/\Delta} = \lim_{\Delta \to 0} \exp\left(\log\left((1-\Delta\rho)^{t/\Delta}\right)\right)$$

$$= \exp\left(\lim_{\Delta \to 0} \log\left((1-\Delta\rho)^{t/\Delta}\right)\right)$$

α s

L'Hôpital

$$= \exp\left(\lim_{\Delta \to 0} \frac{\log(1-\Delta\rho)}{\Delta/t}\right)$$

$$= \exp\left(\lim_{\Delta \to 0} \frac{\frac{-\rho}{1-\Delta\rho}}{1/t}\right) = \exp(-\rho t)$$

Ramsey

$c(t) \rightarrow u(0) / U$ $\qquad$ $C(t) = c(t) \cdot L(t)$

$a(t) / A(t)$ $\qquad$ $A(t) = a(t) \cdot L(t)$

$$\rightarrow \quad \dot{A}(t) = r(t)A(t) + w(t) \cdot L(t) - c(t) \cdot L(t)$$

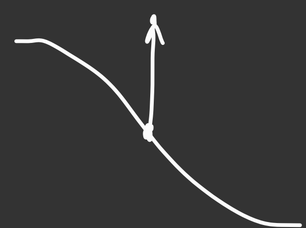
$$\rightarrow \quad \frac{\dot{a}}{a} = \frac{\dot{A}}{A} - \frac{\dot{L}}{L} = r + \frac{wL}{A} - \frac{cL}{A} - n$$

$$\frac{\dot{a}}{a} = r - n + \frac{w}{a} - \frac{c}{a}$$

$$\rightarrow \quad \dot{a} = (r-n)a + w - c$$

Optimization

$$\max_{[a(t), c(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp(-(\rho - n)t) u(c(t)) dt$$



s.t. $\quad \dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t)$

Problem

$$\max_{x(t), y(t)} \int_0^{\infty} \exp(-\rho t) f(x(t), y(t)) dt$$

s.t. $\quad \dot{x}(t) = g(t, x(t), y(t))$

and $\quad x(0) = x_0 \quad$ and $\quad \lim_{t \to \infty} b(t)x(t) \geq x_1$

Hamiltonian

$$H(t, x, y, \lambda) = \exp(-\rho t) f(x, y) + \lambda g(t, x, y)$$

$$\rightarrow \hat{H}(t, x, y, \lambda) = f(x, y) + \mu g(t, x, y)$$

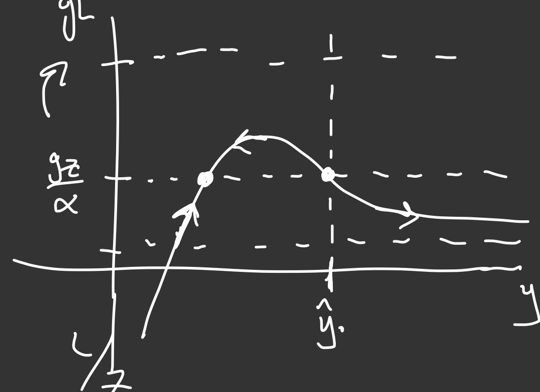
opt. conditions (x, y, μ)

$$H(t,x,y,\mu) = f(x,y) + \mu g(t,x,y)$$

$\to y$ ① $H_y(t, x(t), y(t), \mu(t)) = 0 \quad \forall t$

μ ② $\rho\mu(t) - \dot{\mu}(t) = H_x(t, x(t), y(t), \mu(t))$

assumed
$\to x$ ③ $\dot{x}(t) = g(t, x(t), y(t))$



Malthus

$$\dot{Z} = \eta L \to g_Z = \frac{\dot{Z}}{Z} = \frac{\eta L}{Z} \qquad g_L = g_Z$$

$$n_1 = g_L = g_Z = \frac{\eta L}{Z}$$

$$\frac{L}{Z} = \frac{n_1}{\eta} \;\Big|\; \frac{Z}{L} = \frac{\eta}{n_1}$$

$y \quad L/Z$

$\uparrow \quad \uparrow$

$$\frac{\partial Y}{\partial K}$$

$$r = \frac{\partial Y}{\partial K} = \alpha Z K^{\alpha-1} L^{1-\alpha} \qquad \frac{\frac{\partial Y}{\partial K}}{Y} = \frac{\alpha}{K}$$

$$\uparrow r = \frac{\alpha Y}{K} \leftarrow$$

$$w = \frac{\partial Y}{\partial L} = (1-\alpha) Z K^{\alpha} L^{-\alpha} = \frac{(1-\alpha)Y}{L} \leftarrow$$

$\uparrow$

$$\frac{r}{w} = \frac{\alpha}{1-\alpha} \cdot \frac{L}{K} \uparrow \qquad \frac{\partial Y}{\partial Z} = K^{\alpha} L^{1-\alpha} = \frac{Y}{Z}$$

$$\frac{\partial Y/\partial Z}{w} = \frac{Y/Z}{\frac{(1-\alpha)Y}{L}} = \frac{1}{1-\alpha} \cdot \frac{L}{Z} = \frac{1}{1-\alpha} \cdot \frac{n_1}{\eta} \leftarrow$$

Interpeting μ $\quad \to H_x(t+s) = H_x(t+s, x(t+s), y(t+s), \mu(t+s))$

② $\rho\mu(t) - \dot{\mu}(t) = H_x(t, x(t), y(t), \mu(t))$

asserting ↓

$$\to \mu(t) = \int_0^\infty \exp(-\rho s)\, H_x(t+s)\, ds$$

↓

$$\dot{\mu}(t) = \int_0^\infty \exp(-\rho s)\, \dot{H}_x(t+s)\, ds$$

$$= \left[\exp(-\rho s)\, H_x(t+s)\right]_0^\infty - \int_0^\infty (-\rho)\cdot \exp(-\rho s)\, H_x(t+s)\, ds$$

$$\dot{\mu}(t) = H_x(t) + \rho\, \mu(t)$$

$$\to \rho\mu(t) - \dot{\mu}(t) = H_x(t)$$

Ramsey

$$\max_{a(t),c(t)} \int_0^\infty \exp(-\widetilde{(\rho-n)}t)\, u(c(t))\, dt$$

s.t. $\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t)$ ←

and $a(0) = a_0$

$$H(\overset{t}{a}, c, \mu) = u(c) + \mu\cdot\left[(\overset{\downarrow}{r} - n)a + \overset{\downarrow}{w} - c\right]$$

① $0 = H_c = u'(c) - \mu \to \boxed{\mu = u'(c)}$ $\forall t$

② $(\rho - n)\mu - \dot{\mu} = H_a = \mu(r-n)$

$$\boxed{\dot{\mu} = (\rho - r)\mu}$$

③ $\dot{a} = (r-n)a + w - c$

Transversality

$$\lim_{t\to\infty} \exp(-\rho t) H(t, x(t), y(t), \mu(t)) = 0$$

Assume f and g weakly monotone

$$\lim_{t\to\infty} \exp(-\rho t)\mu(t)x(t) = 0 \qquad \lim_{t\to\infty} \exp(-(\rho-n)t)\mu(t)a(t) = 0$$

$$\left.\begin{aligned} \mu &= u'(c) \\ \dot{\mu} &= (\rho - r)\mu \end{aligned}\right\} \quad \dot{\mu} = u''(c)\cdot\dot{c}$$

$$\rightarrow (\rho-r)\mu = \dot{\mu} = u''(c)\cdot\dot{c}$$

$$(\rho-r)u'(c) = u''(c)\dot{c}$$

$$\rightarrow \dot{c} = (\rho-r)\cdot\frac{u'(c)}{u''(c)}$$

$$\rightarrow \frac{\dot{c}}{c} = (\rho-r)\cdot\underbrace{\frac{u'(c)}{c\cdot u''(c)}} \qquad \varepsilon_u(c) \equiv -\frac{c\cdot u''(c)}{u'(c)}$$

$$\rightarrow \frac{\dot{c}}{c} = -(\rho-r)\cdot\frac{1}{\varepsilon_u(c)} = (r-\rho)\cdot\frac{1}{\varepsilon_u(c)}$$

CRRA

$$u(c) = \frac{c^{1-\theta}-1}{1-\theta}$$

$$u'(c) = c^{-\theta}$$

$$u''(c) = -\theta c^{-\theta-1}$$

$$\varepsilon_u(c) = -\frac{c\cdot(-\theta c^{-\theta-1})}{c^{-\theta}} = \theta$$

$$\text{CRRA} \rightarrow \left\{\begin{aligned} \frac{\dot{c}}{c} &= \frac{r-\rho}{\theta} \\ \dot{a} &= (r-n)a + w - c \end{aligned}\right.$$

Equilibrium

$$\rightarrow r = R - \delta \;\Big|\; R = r + \delta$$

$$w = \frac{\partial y}{\partial L}$$

$$R = \frac{\partial y}{\partial K}$$

Mkt. Clearing

$$a = k$$

$c(0)$? $\quad \frac{\dot c}{c} = \frac{r-\rho}{\theta}$

$R = \frac{\partial Y}{\partial K} = f'(k) \qquad r = R - \delta = f'(k) - \delta$

$w = \frac{\partial Y}{\partial L} = f(k) - k f'(k)$

$a(0) = a_0 \quad \dot a = (r-n)a + w - c$

$a = k \to \dot a = \dot k$

guess $\quad c(0) \to c(t) \to a(t) \to a_\infty = 0$

$$\to \quad \frac{\dot c}{c} = \frac{f'(k) - \delta - \rho}{\theta} = \frac{f'(k) - (\delta + \rho)}{\theta}$$

$$\dot k = (f'(k) - \delta - n)k + f(k) - k f'(k) - c$$

$$\begin{cases} \dot k = f(k) - (\delta + n)k - c \\ \dot c = \frac{1}{\theta} c \cdot \left(f'(k) - (\delta + \rho) \right) \end{cases} \qquad (k, c)$$

Steady state

c^*, k^*

$$\dot k = 0 \to c = \overbrace{f(k)}^{y} - \overbrace{(\delta+n)k}^{i}$$

$$\dot c = 0 \to f'(k) = \delta + \rho$$

$r + \delta$

$\shortparallel$

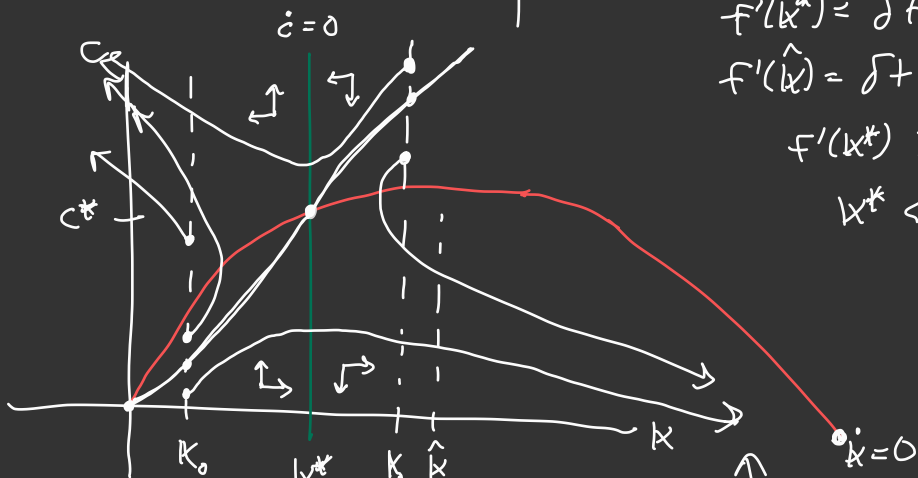
$R = \delta + \rho \quad \to \quad r = \rho$

$f'(k^*) = \delta + \rho$

$f'(\hat k) = \delta + n$

$f'(k^*) > f'(\hat k)$

$k^* < \hat k$



$r > n$?

$\shortparallel$

$R - \delta > n$?

$R > \delta + n$?

$\shortparallel$

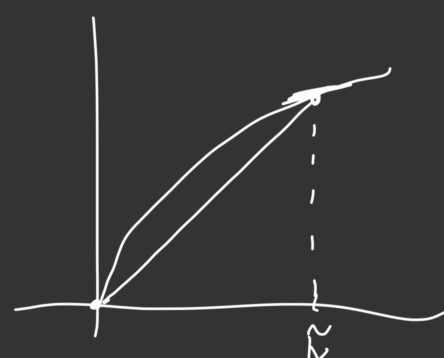
$f'(\tilde k) > \delta + n$?

$f'(\tilde k) < \frac{f(\tilde k)}{\tilde k} = \delta + n$

$\tilde r^* < n$

$f(\tilde k) = (\delta + n)\tilde k$

$\frac{f(\tilde k)}{\tilde k} = \delta + n$



<u>Budget Eq.</u> <u>Lifetime</u>

$$\to \dot{a} = (r-n)a + w - c$$

$$(r-n)a - \dot{a} = c - w$$

<u>Fixed r</u>:

$r \leftrightarrow r-n$ $$a(t) = \int_0^t [w(s) - c(s)] \exp((t-s)r)ds + a_0 \cdot \exp(rt)$$

$$a(t)\exp(-rt) = \int_0^t [w(s) - c(s)] \exp(-sr)ds + a_0$$

$\lim_{t\to\infty}$ $$0 = \int_0^\infty [w(t) - c(t)] \exp(-rt)dt + a_0$$

$$\to \quad a_0 = \int_0^\infty [c(t) - w(t)] \exp(-rt)dt$$

$$\int c(t) \ldots = a_0 + \int w(t) \ldots$$

<u>Time Varying</u> r; $\bar{r}(t) = \frac{1}{t}\int_0^t r(s)\,ds$

$$\frac{\partial(rt)}{\partial t} = r \quad \Bigg| \quad \frac{\partial(\bar{r}(t)t)}{\partial t} = r(t)$$

$$[w(t) - c(t)]\exp(\bar{r}(t)t)$$

$$a_0 = \int_0^\infty [c(t) - w(t)] \exp(-(\bar{r}(t) - n)t)\,dt$$

<u>Ramsey</u> $$H = u(c) + \mu \cdot [(r-n)a + w - c]$$

① $H_c = 0 \to u'(c) - \mu = 0 \to \mu = u'(c) \quad \forall t$

$\quad \hookrightarrow -\frac{\dot{\mu}}{\mu} = -\frac{u''(c)\cdot\dot{c}}{u'(c)}$

② $(\rho - n)\mu - \dot{\mu} = H_a = (r-n)\mu$

$$\to \dot{\mu} = (\rho - r)\mu \to \boxed{-\frac{\dot{\mu}}{\mu} = r - \rho}$$

$$\mu = u'(c) \rightarrow -\frac{\dot\mu}{\mu} = -\frac{u''(c)\,\dot c}{u'(c)} = -\underbrace{\frac{u''(c)\cdot c}{u'(c)}}_{\varepsilon_u} \cdot \underbrace{\frac{\dot c}{c}}_{g_c}$$

$$\dot\mu = (\rho - r)\mu \rightsquigarrow -\frac{\dot\mu}{\mu} = r - \rho$$

$$r - \rho = \varepsilon_u \cdot g_c$$

$$\left[r = \rho + \varepsilon_u \cdot g_c \right.$$

<u>Euler Eqn.</u>

CRRA : $\varepsilon_u = \theta$

$$\rightarrow \frac{\dot c}{c} = \frac{r-\rho}{\varepsilon_u} \rightarrow \frac{d\log(c)}{dt} = \frac{r^{(t)}-\rho}{\varepsilon_u(c)} = \frac{r^{(t)}-\rho}{\theta}$$

$$\rightarrow \log(c(t)) = \log(c(0)) + \int_0^t \frac{r(s)-\rho}{\theta}\,ds$$

$$\rightarrow \log(c(t)) = \log(c(0)) + \left(\frac{\bar r(t)-\rho}{\theta}\right)t$$

$$\rightarrow \log\left(\frac{c(t)}{c(0)}\right) = \left(\frac{\bar r(t)-\rho}{\theta}\right)t$$

$$\rightarrow c(t) = c(0)\cdot \exp\left(\frac{\bar r(t)-\rho}{\theta}\cdot t\right)$$

<u>Transversality Condition</u>

$$\lim_{t\to\infty} \exp(-(\rho-n)t)\,\mu(t)\,a(t) = 0$$

$$\frac{\dot\mu}{\mu} = -(r-\rho)$$

$$\lim_{t\to\infty} \exp(-(\bar r(t)-n)t)\,a(t) = 0$$

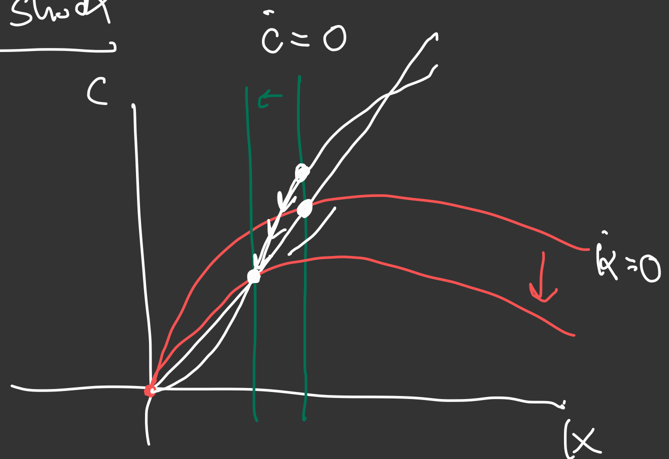
$$\frac{d\log\mu}{dt} = -(r-\rho)$$

$$\mu(t) = \mu(0)\exp(-(\bar r(t)-\rho)t)$$

$$\bar r(t) > n$$

$$r^* = \lim_{t\to\infty} r(t) > n \qquad r^* = \rho$$

z shock



$$\to \frac{\dot c}{c} = \frac{f'(k)-\delta-\rho}{\theta}$$

$$\dot k = f(k) - (\delta+n)k - c$$

$$\to f(k) = z k^{\alpha}$$

$$f'(k) = \alpha z k^{\alpha-1}$$

$$\dot c = 0 : \quad f'(k,z) = \delta+\rho$$

$$\dot k = 0 ; \quad c = f(k,z) - (\delta+n)k$$

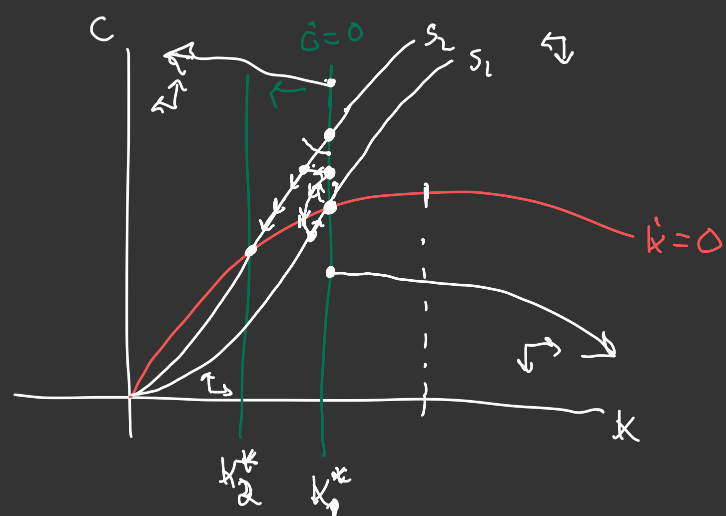
Cap. gains tax (τ)

$$k^* \to \frac{\dot c}{c} = \frac{r-\rho}{\theta} = \frac{(1-\overset{\downarrow}{\tau})(f'(k)-\delta)-\rho}{\theta}$$

$$\to r = (1-\tau)(f'(k)-\delta)$$

$$\dot k = f(k) - (\delta+n)k - c$$

$$\dot k = 0 : \quad c = f(k) - (\delta+n)k$$



Linear

$$\dot x(t) = Ax(t) + b$$

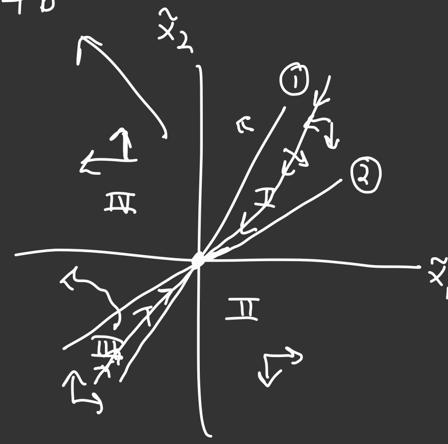
$$\dot x(t) = 0 \to Ax^* \overset{\downarrow}{=} -b \xrightarrow{A^{-1}} \boxed{x^* = -A^{-1}b}$$

$$\tilde x = x - x^* \leftarrow$$

$$\to \dot{\tilde x}(t) = \dot x(t) = Ax(t) + b$$

$$\dot{\tilde x}(t) = A(\tilde x + x^*) + b = A\tilde x + [\cancel{Ax^* + b}]^{0} = A\tilde x(t)$$

$\rightarrow \left[\dot{\tilde{x}} = A\tilde{x} \right. \quad \rightarrow \tilde{x} = x - x^* \quad x^* = A^{-1}b$ | $\rightarrow \dot{x} = Ax + b$



Eigen-stuff

$\rightarrow Av_i = \lambda_i v_i \qquad (v_i, \lambda_i) \qquad V = [v_1 \cdots v_n]$

$AV = V\Lambda \qquad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\rightarrow A = V\Lambda V^{-1} \rightarrow V^{-1}A = \Lambda V^{-1}$

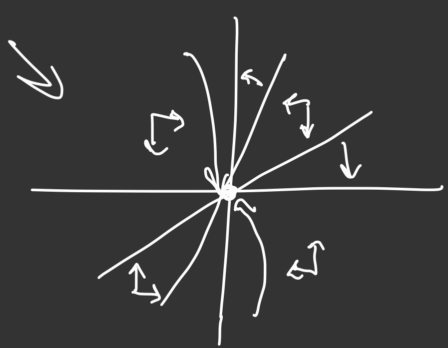
$\rightarrow \hat{x} = V^{-1}\tilde{x} \rightarrow \dot{\hat{x}} = V^{-1}\dot{\tilde{x}} = V^{-1}A\tilde{x} = \Lambda \underbrace{V^{-1}\tilde{x}}_{\hat{x}} = \Lambda\hat{x}$

$\tilde{x} = V\hat{x}$

$\rightarrow \boxed{\dot{\hat{x}} = \Lambda\hat{x}}$

$\rightarrow \dot{\hat{x}}_i(t) = \lambda_i \hat{x}_i(t) \;\; \forall i \rightarrow \hat{x}_i(t) = \hat{x}_i(0) \cdot \exp(\lambda_i t)$

$\lambda_i > 0 \rightarrow \hat{x}_i(0) = 0$



$\dot{\hat{x}}_i = 0 \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \left\{ a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 = 0 \right.$

Linearization:

$\dot{x}(t) = G(x(t)) \qquad A = J(x^*)$

$\rightarrow \dot{x}(t) = J(x^*)(x(t) - x^*)$

Ramsey:

$\dot{k} = f(k) - (n+\delta)k - c$

$\frac{\dot{c}}{c} = \frac{1}{\varepsilon_u(c)} \cdot \left[f'(k) - (\delta + \rho) \right]$

$$\dot{k} = f(k) - (n+\delta)k - c$$

$$\dot{c} = \frac{c}{\varepsilon_u(c)} \cdot \left[f'(k) - (\delta + \rho) \right]$$

$$J_{kk} = f'(k) - (n+\delta) \overset{*}{=} (\delta+\rho) - (n+\delta) = \rho - n = \tilde{\rho} > 0$$

$$J_{kc} = -1$$

$$J_{ck} = \frac{c}{\varepsilon_u(c)} \cdot f''(k) = -Q \qquad J^* = \begin{bmatrix} \tilde{\rho} & -1 \\ -Q & 0 \end{bmatrix}$$

$$J_{cc} = 0$$

$$\det(J^*) = -Q < 0$$

$$\det(A - \lambda I) = 0 \quad \Bigg| \quad 0 = \begin{vmatrix} \tilde{\rho} - \lambda & -1 \\ -Q & -\lambda \end{vmatrix} = -\lambda(\tilde{\rho} - \lambda) - Q = 0$$

$$\lambda(\tilde{\rho} - \lambda) = -Q$$

<u>Eigenvalues</u>

$$\lambda^2 - \tilde{\rho}\lambda - Q = 0$$

$$\det(A) = \det(V \Lambda V^{-1})$$

$$= \det(V)\det(\Lambda)\det(V^{-1})$$

$$= \det(\Lambda) = \prod_{i=1}^{n} \lambda_i$$

$$\lambda_{1/2} = \lambda_{\pm} = \frac{\tilde{\rho} \pm \sqrt{\tilde{\rho}^2 + 4Q}}{2}$$

$$\lambda_1 > 0$$

$$\lambda_2 < 0$$

$$\rightarrow \lambda_1 \lambda_2 = -Q < 0 \quad \Big| \quad \lambda_1 + \lambda_2 = \tilde{\rho} \quad \Big| \longrightarrow \lambda_1(\tilde{\rho} - \lambda_1) = -Q$$

$$\operatorname{tr}(A) = \operatorname{tr}(V \Lambda V^{-1})$$

$$= \operatorname{tr}(\Lambda V^{-1} V) = \operatorname{tr}(\Lambda) = \sum_{i=1}^{n} \lambda_i$$

$$\hat{x}_1(0) = 0 \rightarrow \hat{x}_1(t) = 0$$

$$\hat{x}_2(0) = z \rightarrow \hat{x}_2(t) = z \exp(\lambda_2 t)$$

Eigenvectors

$$\lambda_{1/2} = \frac{\tilde{p} \pm \sqrt{\tilde{p}^2 + 4Q}}{2}$$

$$A v_i = \lambda_i v_i \rightarrow \begin{bmatrix} \tilde{p} & -1 \\ -Q & 0 \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \end{bmatrix} = \lambda_i \begin{bmatrix} v_1^i \\ v_2^i \end{bmatrix}$$

$$\tilde{p} v_1^i - v_2^i = \lambda_i v_1^i$$

$$\rightarrow -Q v_1^i = \lambda_i v_2^i$$

$$\boxed{\frac{v_1^i}{v_2^i} = -\frac{\lambda_i}{Q}}$$

$$v^i = \begin{bmatrix} \lambda_i \\ -Q \end{bmatrix} \rightarrow V = \begin{bmatrix} \lambda_1 & \lambda_2 \\ -Q & -Q \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 0 \\ z \end{bmatrix} \qquad \begin{bmatrix} \tilde{k} \\ \tilde{c} \end{bmatrix} = \overbrace{\tilde{x} = V\hat{x}} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ -Q & -Q \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix}$$

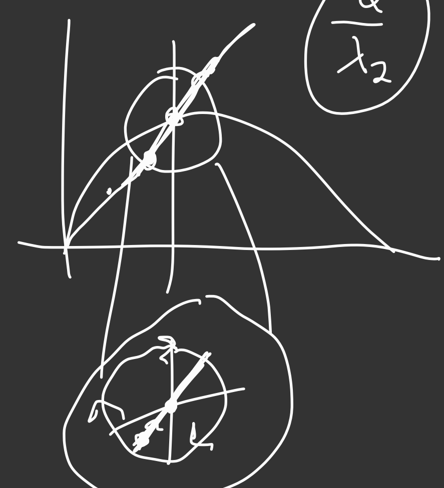
$$\tilde{k}(0) = k_0$$

$$z(0) = \frac{\tilde{k}(0)}{\lambda_2} = \frac{k_0}{\lambda_2}$$

$$\tilde{c}(0) = -Q z(0) = -\frac{Q}{\lambda_2} \cdot k_0$$

$$\begin{bmatrix} \tilde{k} \\ \tilde{c} \end{bmatrix} = \begin{bmatrix} \lambda_2 z \\ -Q z \end{bmatrix}$$

$$\frac{-Q}{\lambda_2} > 0$$



$$\hat{x}(0) = \begin{bmatrix} 0 \\ z(0) \end{bmatrix} = V^{-1} \tilde{x}(0) = V^{-1} \begin{bmatrix} \tilde{k}(0) \\ \tilde{c}(0) \end{bmatrix}$$

$$D = \sqrt{\tilde{p}^2 + 4Q}$$

$$V^{-1} = \frac{1}{Q(\lambda_2 - \lambda_1)} \begin{bmatrix} -Q & -\lambda_2 \\ Q & \lambda_1 \end{bmatrix}$$

$$V^{-1} = \frac{1}{-QD} \begin{bmatrix} -Q & -\lambda_2 \\ Q & \lambda_1 \end{bmatrix} \rightarrow \hat{x} = \frac{-1}{QD} \begin{bmatrix} -Q\tilde{k} - \lambda_2 \tilde{c} \\ Q\tilde{k} + \lambda_1 \tilde{c} \end{bmatrix} = \begin{bmatrix} 0 \\ z \end{bmatrix}$$

$$z = \left[Q\tilde{k} + \lambda_1 \cdot \frac{-Q}{\lambda_2} \tilde{k} \right] \frac{-1}{QD} = \frac{\frac{\lambda_1}{\lambda_2} - 1}{D} \tilde{k} \rightarrow \boxed{\tilde{c} = \frac{-Q}{\lambda_2} \cdot \tilde{k}}$$

$$z = \frac{\frac{\lambda_1}{\lambda_2} - 1}{D} \tilde{k} = \frac{1}{\lambda_2} \cdot \frac{\lambda_1 - \lambda_2}{D} \tilde{k} = \frac{1}{\lambda_2} \cdot \tilde{k}$$

$$z(t) = \exp(\lambda_2 t) z(0) = \exp(\lambda_2 t) \cdot \frac{1}{\lambda_2} \cdot \tilde{k}(0)$$

$$\rightarrow \tilde{k}(t) = \tilde{k}(0) \exp(\lambda_2 t)$$

$$\tilde{c}(t) = \tilde{k}(0) \cdot \frac{a}{-\lambda_2} \exp(\lambda_2 t)$$

<u>Long-term growth</u>

$$\frac{\dot{c}}{c} = \frac{1}{\varepsilon_u(c)} \cdot (r - \rho) = \frac{r-\rho}{\theta}$$

$$g_c = \frac{r-\rho}{\theta} \rightarrow r = \rho + \theta g_c \overset{*}{=} \rho + \theta g$$

$$g_c = g = g_A$$

<u>Factor Prices</u> r, w

$$R = F_K(K, AL) = F_K\left(\frac{K}{AL}, 1\right)$$

$$= F_K(\tilde{k}, 1) = f'(\tilde{k})$$

$$r = R - \delta = f'(\tilde{k}) - \delta$$

$$\rho + \theta g = r = f'(k) - \delta$$

$$\rightarrow f'(k^*) = \rho + \delta + \theta g$$

$$\lim_{c \to \infty} \varepsilon_{u'}(c) = \varepsilon_{u'}^*$$

$$w = F_L(K, AL)$$

$$= A\left[f(\tilde{k}) - \tilde{k} f'(\tilde{k})\right]$$

$$\tilde{w} \equiv \frac{w}{A} = f(\tilde{k}) - \tilde{k} f'(\tilde{k})$$

$$k = \frac{K}{L} \qquad y = \frac{Y}{L}$$

$$\tilde{k} = \frac{K}{AL} \qquad \tilde{y} = \frac{Y}{AL}$$

$$Y = K^\alpha (AL)^{1-\alpha}$$

$$\frac{Y}{AL} = \left(\frac{K}{AL}\right)^\alpha$$

$$\tilde{y} = \tilde{k}^\alpha$$

$$Y = F(K, AL)$$

$$\rightarrow \frac{Y}{AL} = F\left(\frac{K}{AL}, 1\right)$$

$$\tilde{y} = F(\tilde{k}, 1) \equiv f(\tilde{k})$$

$$\left[f(x) \equiv F(x, 1) \rightarrow \underset{F_1(x,1)}{\overset{f'(x)}{\shortparallel}} \right.$$

$$F(K, AL) = K F_K(K, AL) + L F_L(K, AL)$$

$$f(\tilde{k}) = \tilde{k} f'(\tilde{k}) + \frac{1}{A} \cdot F_L(K, AL)$$

$$\dot{k} = f(k) - (\delta+n)k - c$$

$$\dot{\tilde{k}} = f(\tilde{k}) - (\delta+n+g)\tilde{k} - \tilde{c}$$

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{k}}{k} - g \Rightarrow \dot{\tilde{k}} = \dot{k}\cdot\frac{\tilde{k}}{k} - g\tilde{k} = \frac{\dot{k}}{A} - g\tilde{k}$$

Fully normalized $\quad \tilde{c} = \frac{c}{A} \to c = A\cdot\tilde{c} \qquad A(t) = A_0\cdot\exp(gt) \equiv \exp(gt)$

$$u = \int_0^\infty \left[\frac{c_{(t)}^{1-\theta}-1}{1-\theta}\right]\exp(-(\rho-n)t)\,dt$$

$$= \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta}\cdot\exp(-(\rho-n)t)\,dt - \frac{1}{1-\theta}\cdot\frac{1}{\rho-n}$$

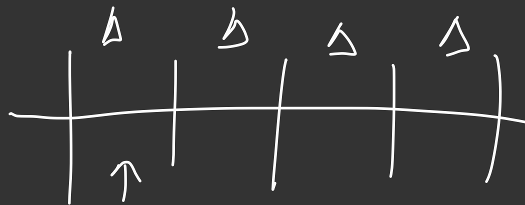
$$\to = \int_0^\infty \frac{[A\tilde{c}]^{1-\theta}}{1-\theta}\cdot\exp(-(\rho-n)t)\,dt - Z$$

$$= \int_0^\infty \frac{\tilde{c}^{1-\theta}}{1-\theta}\exp(-(\overbrace{\rho-n-(1-\theta)g}^{\tilde{\rho}})t)\,dt - Z$$

$u(\tilde{c}) \qquad (\tilde{c}, \tilde{k})$

Value functions

β - discount $\quad \beta\in[0,1]$



$\updownarrow$

$\rho \qquad \beta = \exp(-\rho\Delta) \qquad y = f(k) = c+i \qquad \dot{k} = i-\delta k$

$\Delta\to 0$

$$(*)\quad V(k,t) = \max_i\, \Delta\cdot u(\overbrace{f(k)-i}^{c}) + \exp(-\rho\Delta)\cdot V(k+\Delta\cdot(i-\delta k), t+\Delta)$$

$$V(k,t) = \Delta\cdot u(f(k)-i) + V(k,t)$$
$$+\Delta\left[-\rho\cdot V(k,t) + V_k(k,t)\cdot(i-\delta k) + \dot{V}(k,t)\right] + o(\Delta^2)$$

$$0 = u(f(k) - i) - \rho v(k,t) + \left[(i - \delta k) v_k(k,t) + \dot{v}(k,t)\right] \qquad \frac{o(\Delta^2)}{\Delta} \sim o(\Delta)$$

$$\downarrow$$

$$\cancel{0}$$

$$\rightarrow ① \ \rho v - \dot{v} = u(f(k) - i) + (i - \delta k) v_k$$

$$i(k) \qquad i_k = \frac{di}{dk}$$

$$\rho v = u(c) + \left[(i - \delta k) v_k + \dot{v}\right]$$

FOC$_i$: $- \Delta \cdot u'(f(k) - i) + \exp(-\Delta\rho) \cdot v_k(k + \Delta \cdot (i - \delta k), t + \Delta)\, \Delta = 0$

$$u'(f(k) - i) = \exp(-\Delta\rho)\, v_k(k + \Delta(i - \delta k), t + \Delta)$$

$$u'(f(k) - i) = v_k(k,t)$$

$$\downarrow$$

$$② \ u'(c) = v_k \qquad \dot{k} = i - \delta k$$

EC: $\rho v_k - \dot{v}_k = u'(c) \cdot f'(k) + (i - \delta k) v_{kk} - \delta v_k$

$$+ i_k \cdot \left[\underbrace{-u'(c) + v_k}_{0}\right]$$

steady state: $\dot{v} = 0$, $\dot{k} = 0$

$$\rho v_k = u'(c) \cdot f'(k) - \delta v_k$$

$$\rho = f'(k) - \delta \rightarrow \begin{cases} f'(k^*) = \rho + \delta \\ c^* = f(k^*) - \delta k^* \end{cases}$$

Climate Extern.

state: k, S

$\rightarrow \dot{k} = y - \delta k - c$

$\dot{S} = ek - \tau S$

$y = \overbrace{(1 - \gamma S)}^{d} \cdot f(k)$ choice: c

$+ f'(k)$

$- e \cdot \gamma \cdot f(k)$

$$H = u(c) + \mu \cdot \left[(1 - \gamma S) f(k) - \delta k - c\right] + \lambda \cdot \left[ek - \tau S\right]$$

$$H = u(c) + \mu\cdot\left[(1-\gamma s)f(k) - \delta k - c\right] + \lambda\cdot\left[ek - \tau s\right]$$

① $0 = H_c = u'(c) - \mu \rightarrow \boxed{\mu = u'(c)}$

② $\rho\mu - \dot{\mu} = H_k = \mu\cdot\left[(1-\gamma s)f'(k) - \delta\right] + \lambda\cdot e$

③ $\rho\lambda - \dot{\lambda} = H_s = -\mu\gamma f(k) - \lambda\tau$

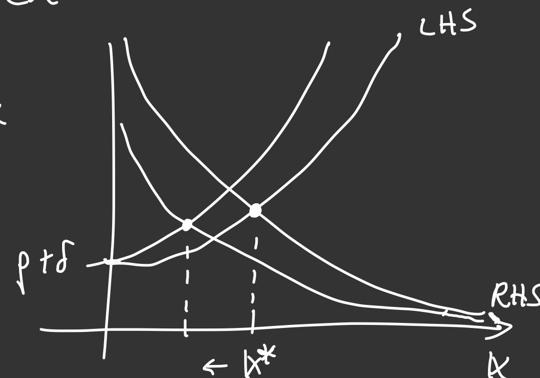
② $\rightarrow \dfrac{\dot{\mu}}{\mu} = -(1-\gamma s)f'(k) + \delta + \rho - e\cdot\dfrac{\lambda}{\mu}$

③ $\rightarrow \dfrac{\dot{\lambda}}{\lambda} = \gamma f(k)\cdot\dfrac{\mu}{\lambda} + \tau + \rho$ $\qquad d = 1-\gamma s$

$x \equiv \dfrac{\lambda}{\mu} \rightarrow \dfrac{\dot{x}}{x} = \dfrac{\dot{\lambda}}{\lambda} - \dfrac{\dot{\mu}}{\mu} = \gamma f(k)\cdot\dfrac{1}{x} + \tau + \rho + d f'(k) - \delta - \rho + e\cdot x$

$= (\tau - \delta) + \gamma f(k)\cdot\dfrac{1}{x} + d f'(k) + e\cdot x$

$\boxed{\tau = \delta} \rightarrow \dfrac{\dot{x}}{x} = \gamma f(k)\cdot\dfrac{1}{x} + d f'(k) + ex$



Steady State: ② $\rightarrow 0 = -d f'(k) + \delta + \rho - e\cdot x$

③ $\rightarrow 0 = \gamma f(k)\cdot\dfrac{1}{x} + \tau + \rho$

$\dot{s} = ek - \tau s$

$\rightarrow x = -\dfrac{\gamma f(k)}{\tau + \rho}$

$ek = \tau s$

$s = \dfrac{e}{\tau}\cdot k$

$\rightarrow 0 = -d f'(k) + \delta + \rho + \dfrac{e\gamma f(k)}{\tau + \rho}$

$\downarrow$

$\rho + \delta + \dfrac{e\gamma f(k)}{\rho + \tau} = d f'(k) = (1-\gamma s)f'(k) = \left(1 - \dfrac{e\gamma k}{\tau}\right)f'(k)$

LHS $\qquad$ RHS

$\boxed{\alpha = 1}$ $\quad f(k) = A\cdot k \quad f'(k) = A$

① $U = \int_0^\infty u(c(t)) \exp(-\rho t)\, dt$

② $U = \int_0^\infty u(c(t))\, L(t) \exp(-\rho t)\, dt$

$= \int_0^\infty u(c(t)) \exp(-(\rho - n)t)\, dt$

③ $U = \int_0^\infty \left[\frac{c(t)^{1-\theta} - 1}{1-\theta} \right] \exp(-(\rho - n)t)\, dt$ $\qquad \tilde{c} = \frac{c}{A} = \frac{C}{AL}$

$= \int_0^\infty \left[\frac{[A(t)\tilde{c}(t)]^{1-\theta} - 1}{1-\theta} \right] \exp(-(\rho-n)t)\, dt \rightarrow \underline{c = A\tilde{c}}$

$= \int_0^\infty [A(t)\tilde{c}(t)]^{1-\theta} \exp(-(\rho-n)t)\, dt$

$= \int_0^\infty \tilde{c}(t)^{1-\theta} [\exp(gt)]^{1-\theta} \exp(-(\rho-n)t)\, dt$

$\rightarrow = \int_0^\infty \tilde{c}(t)^{1-\theta} \exp(-\overbrace{(\rho - n - (1-\theta)g)}\, t)$

$\phi \in (0,1)$

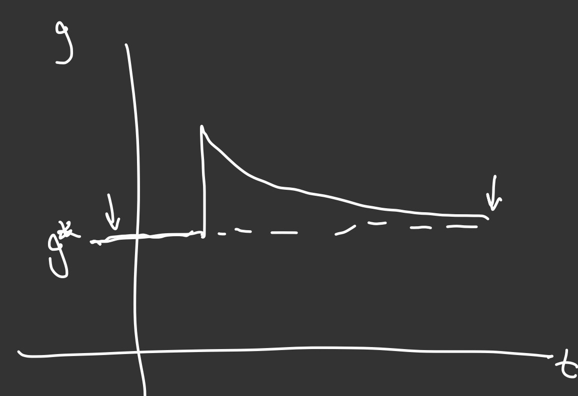
$g = \frac{\dot{A}}{A} = \frac{\gamma R^\eta}{A^{1-\phi}}$

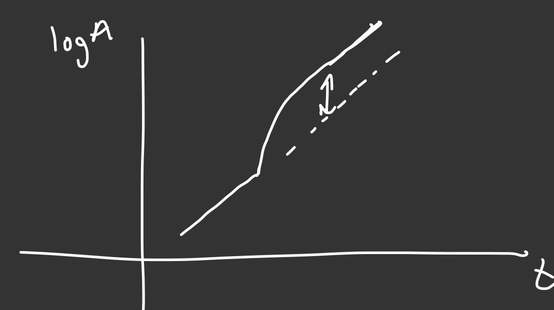
$\eta \cdot n = (1-\phi) g$

$\rightarrow \boxed{g^* = \frac{\eta \cdot n}{1-\phi}}$

$n = \frac{\dot{L}}{L} \rightarrow \frac{\dot{R}}{R} = n$

$R = SL$





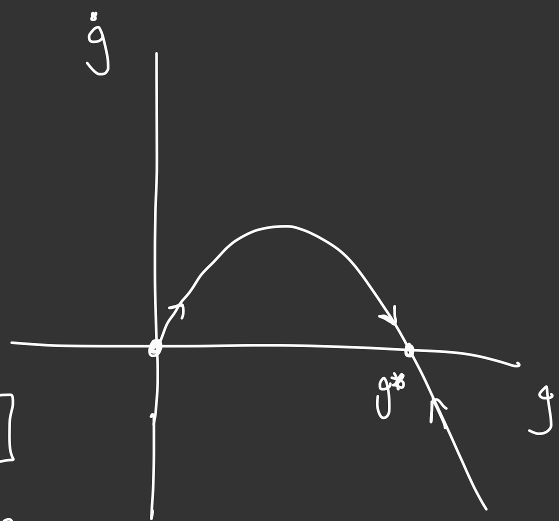
$\underline{\phi = 1}$ $\quad g = \gamma K^{\eta} = \gamma (sL)^{\eta}$

$\underline{\text{Dynamics}}$

$$g = \frac{\gamma K^{\eta}}{A^{1-\phi}} \rightarrow \frac{\dot{g}}{g} = \eta \cdot n - (1-\phi) g$$

$$\rightarrow \dot{g} = g \cdot [\eta n - (1-\phi) g]$$

$$= (1-\phi) g \cdot \left[\frac{\eta n}{1-\phi} - g\right] = (1-\phi) g (g^* - g)$$

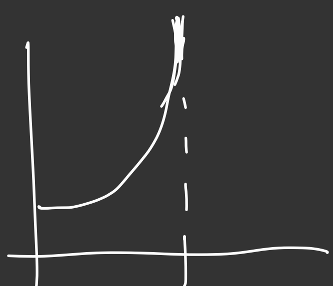


$\phi > 1$, $n = 0$

$$\frac{dg}{dt} = \dot{g} = (\phi - 1) g^2$$

$$\frac{dg}{g^2} = (\phi-1) dt \rightarrow -\frac{1}{g(t)} = (\phi-1)t + C$$

$$g(0) = g_0 \quad C = -\frac{1}{g_0} \rightarrow \frac{-1}{g(t)} = (\phi-1)t - \frac{1}{g_0}$$



$$\rightarrow g(t) = \frac{1}{\frac{1}{g_0} - (\phi-1)t} = \frac{g_0}{1-(\phi-1)g_0 \cdot t}$$

$$\frac{d \log A}{dt} = \frac{\dot{A}}{A} = \frac{g_0}{1-(\phi-1)g_0 t}$$

$$\log A = \frac{-1}{\phi-1}\Big[\log\big(1-(\phi-1)g_0 t\big)\Big]_0^{t^*} = \infty$$

Endog. Growth Models

$n = 0$

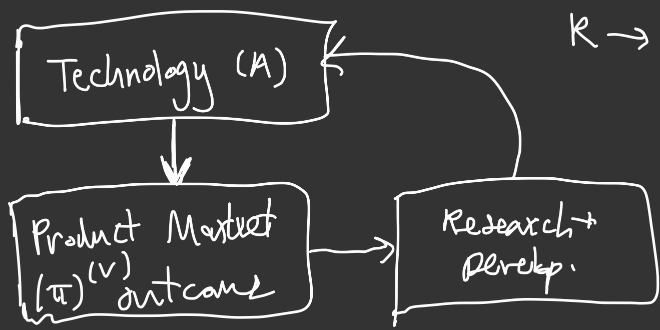
$L(t) = 1$

y - output

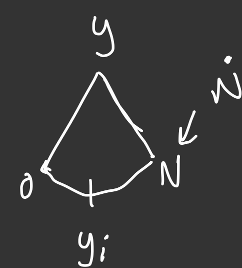
R - researchers

P - producers

$$\left.\begin{array}{l} R \\ P \end{array}\right\} 1 = P + R$$



$R \rightarrow \dot{N}$



Romer

$$y = \left[\int_0^N y_i^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} \qquad y_i = l_i$$

$$y_i = l_i = \frac{P}{N} \rightarrow y = \left[\int_0^N \left(\frac{P}{N}\right)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} = \frac{P}{N}\cdot\left[\int_0^N di\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$= \frac{P}{N}\cdot N^{\frac{\varepsilon}{\varepsilon-1}} = \boxed{P\cdot N^{\frac{1}{\varepsilon-1}} = y}$$

$$\rightarrow g \equiv \frac{\dot{N}}{N} = g_N \rightarrow \boxed{g_y = \frac{g}{\varepsilon-1}}$$

Aggregator Firm

$p_i \rightarrow y_i, y$

$$\pi_i = p_i y_i - w l_i$$

$$= p_i y_i - w y_i = (p_i - w)\cdot y_i$$

$$y_i \quad p_i$$

$$\Pi = \left[\int_0^N y_i^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^N p_i y_i di$$

$$\frac{\partial \Pi}{\partial y_i} = \left(\frac{\cancel{\varepsilon}}{\cancel{\varepsilon-1}}\right)\left[\int_0^N y_i^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{1}{\varepsilon-1}}\cdot\left(\frac{\cancel{\varepsilon-1}}{\cancel{\varepsilon}}\right) y_i^{-1/\varepsilon} - p_i = 0$$

$$= y^{1/\varepsilon} y_i^{-1/\varepsilon} - p_i = 0 \qquad \varepsilon = 1 \rightarrow p_i y_i = y$$

$$\rightarrow p_i = \left(\frac{y}{y_i}\right)^{1/\varepsilon} \rightarrow y_i = y\cdot p_i^{-\varepsilon}$$

Producer i $\quad y_i = l_i \quad p_i = \left(\frac{y_i}{y}\right)^{-1/\varepsilon} \quad y_i = y \cdot p_i^{-\varepsilon}$

$$\pi_i = p_i y_i - w l_i = (p_i - w)\cdot y_i$$

$$\pi(p) = (p - w)\cdot y(p)$$

$$\pi(y) = (p(y) - w)\cdot y$$

$$\varepsilon_p = \frac{1}{\varepsilon}$$

$$\varepsilon_y = -\frac{y'(p)\cdot p}{y(p)} = \varepsilon$$

$$\frac{\partial \pi}{\partial y} = (p(y) - w) + p'(y)\cdot y = 0$$

$$\rightarrow p(y) - w = -p'(y)\cdot y$$

$$\rightarrow 1 - \frac{w}{p} = \frac{p(y) - w}{p(y)} = -\frac{p'(y)\cdot y}{p(y)} \equiv \varepsilon_p > 0$$

$$\rightarrow \boxed{\frac{p}{w} = \frac{1}{1-\varepsilon_p}} = \frac{1}{1 - 1/\varepsilon} = \frac{\varepsilon}{\varepsilon - 1}$$

Price Markup

$$\frac{p_i}{w} = \frac{\varepsilon}{\varepsilon-1} \rightarrow p_i = w\left(\frac{\varepsilon}{\varepsilon-1}\right)$$

$$l_i = \rightarrow y_i = y\cdot\left[w\left(\frac{\varepsilon}{\varepsilon-1}\right)\right]^{-\varepsilon}$$

$$\pi_i = (p_i - w)\cdot y_i$$

$$\pi_i = \left(\frac{w}{\varepsilon-1}\right) y \cdot \left[w\left(\frac{\varepsilon}{\varepsilon-1}\right)\right]^{-\varepsilon}$$

$$\pi_i = y\cdot\frac{1}{\varepsilon}\cdot\left[w\left(\frac{\varepsilon}{\varepsilon-1}\right)\right]^{1-\varepsilon}$$

$$\dot{N} = \gamma \underset{\phi=1}{N} \underset{\eta=1}{R}$$

$$\frac{\dot{N}}{N} = \gamma R$$

Aggregates $\rightarrow 1 = P + R \quad \underset{\parallel}{y_i}$

$$P \equiv \int_0^N l_i di = N l_i \rightarrow l_i = \frac{P}{N}$$

$$\pi_i = (p_i - w)\cdot y_i$$

$$\rightarrow \pi = \left(\frac{w}{\varepsilon-1}\right)\cdot\frac{P}{N} = \frac{wP}{(\varepsilon-1)N}$$

$$\rightarrow \boxed{\frac{N\pi}{wP} = \frac{1}{\varepsilon-1}}$$

$$\pi = \frac{wP}{(\varepsilon-1)N}$$

$$\pi \rightarrow \underset{\text{NPV}}{V} \overset{=}{\longleftrightarrow} w.$$

w/ $\quad$ free entry

Values

ops

No arbitrage

borrow $\$v(t)$

buy 1 share

$\pi(t)$

$r(t)$

$\downarrow$

$v(t)$

$$0 = +v(t) - v(t) + \Delta\pi(t) + v(t+\Delta) - (1+\Delta r)v(t)$$

$$= \Delta\pi(t) + v(t+\Delta) - (1+\Delta r)v(t)$$

$$0 = \Delta\pi(t) + [v(t+\Delta) - v(t)] - \Delta r\, v(t)$$

$$0 = \pi(t) + \frac{v(t+\Delta) - v(t)}{\Delta} - r v(t)$$

$$\Delta \to 0 \quad \to \quad 0 = \pi(t) + \dot{v}(t) - r v(t)$$

$$\boxed{r v(t) - \dot{v}(t) = \pi(t)}$$

$$r v(t) = \pi(t) + \dot{v}(t)$$

Value

$$r v - \dot{v} = \pi$$

$$\to r - \frac{\dot{v}}{v} = \frac{\pi}{v} \to r - g_v = \frac{\pi}{V} \to V = \frac{\pi}{r - g_v}$$

$$\to g_\pi^* = g_v^* \qquad \to V = \left(\frac{1}{r - g_v}\right) \cdot \frac{wP}{(\varepsilon - 1)\cdot N}$$

Free Entry

$$\dot{N} = \gamma N R \quad \Big| \quad 0 = \dot{N} \cdot V - wR = \gamma N R \cdot V - wR$$

$$= R \cdot [\gamma N V - w]$$

$$\uparrow$$

$$\gamma N V = w$$

$$\frac{\gamma}{r - g_v} \cdot \frac{wP}{\varepsilon - 1} = \rho$$

Last lecture

Demand: $\left(\frac{y_i}{y}\right)^{-1/\varepsilon} = p_i = \left(\frac{\varepsilon}{\varepsilon-1}\right) w$

$\rightarrow$ Profit: $\pi_i = \frac{wP}{(\varepsilon-1)N}$

Value: $v_i = \frac{\pi_i}{r-g_v}$

Euler: $r = \rho + g_c = \rho + g_y$

Free entry: $\underbrace{\gamma N}_{N/R} v_i = w$

output: $y = P \cdot N^{\frac{1}{\varepsilon-1}}$

$\rightarrow c = y$

$c + \dot{a} = ra + \overbrace{w + [N\pi - wR]}^{y}$

$= ra + [wP + wR]$

$+ [y - wP]$

$- wR$

$c + \dot{a} = ra + y$

$\rightarrow c = y \quad + \quad \dot{a} = 0$

$\rightarrow \gamma N \cdot \frac{\pi_i}{r-g_v} = w \rightarrow \frac{\gamma \cancel{N}}{r-g_v} \cdot \frac{wP}{(\varepsilon-1)\cancel{N}} = \cancel{w} \rightarrow \frac{1}{r-g_v} \cdot \frac{\gamma P}{(\varepsilon-1)} = 1$

$\rightarrow r - g_v = \frac{\gamma P}{\varepsilon - 1} \rightarrow \boxed{\rho + g_y - g_v = \frac{\gamma P}{\varepsilon-1}}$

Wages: $\left(\frac{y_i}{y}\right)^{-1/\varepsilon} = \overset{\text{price}}{p} = \left(\frac{\varepsilon}{\varepsilon-1}\right) w$

$\rightarrow l_i = y_i = y \cdot \left[\left(\frac{\varepsilon}{\varepsilon-1}\right) w\right]^{-\varepsilon} = P \cdot N^{\frac{1}{\varepsilon-1}} \left[\left(\frac{\varepsilon}{\varepsilon-1}\right) w\right]^{-\varepsilon}$

$\int_0^N l_i di \rightarrow \cancel{P} = \cancel{P} N^{\frac{\varepsilon}{\varepsilon-1}} \left[\left(\frac{\varepsilon}{\varepsilon-1}\right) w\right]^{-\varepsilon} \rightarrow 1 = N^{\frac{1}{\varepsilon-1}} \left[\left(\frac{\varepsilon}{\varepsilon-1}\right) w\right]^{-1}$

$\rightarrow \boxed{w = \left(\frac{\varepsilon-1}{\varepsilon}\right) N^{\frac{1}{\varepsilon-1}}}$

$\rightarrow g_w = \frac{g}{\varepsilon-1}$

$wP = \left(\frac{\varepsilon-1}{\varepsilon}\right) P \cdot N^{\frac{1}{\varepsilon-1}} = \left(\frac{\varepsilon-1}{\varepsilon}\right) y$

$\frac{wP}{y} = \frac{\varepsilon-1}{\varepsilon} \qquad \frac{N\pi}{y} = \frac{1}{\varepsilon}$ w

$\rightarrow wP + N\pi = y = [wP + wR] + [N\pi - wR]$

$$\boxed{\rho + g_y - g_v = \frac{\gamma p}{\varepsilon - 1}}$$

$$\boxed{\rho + g_p + g = \frac{\gamma p}{\varepsilon - 1}}$$

$$\gamma N v_i = w$$

$$\downarrow$$

$$g + g_v = g_w = \frac{g}{\varepsilon - 1}$$

$$\rightarrow g_v = \left(\frac{2-\varepsilon}{\varepsilon - 1}\right) g$$

$$g_v = \frac{g}{\varepsilon - 1} - g$$

$$\frac{1}{\varepsilon - 1} - 1 = \frac{1 - \varepsilon + 1}{\varepsilon - 1} = \frac{2 - \varepsilon}{\varepsilon - 1}$$

$$y = p N^{\frac{1}{\varepsilon - 1}}$$

$$\downarrow$$

$$g_y = g_p + \frac{g}{\varepsilon - 1}$$

$$\dot{N} = \gamma N R$$

$$\rightarrow g = \frac{\dot{N}}{N} = \gamma R = \gamma(1-p)$$

$$\rightarrow \rho + \frac{\dot{p}}{p} + \gamma(1-p) = \frac{\gamma p}{\varepsilon - 1} \iff \frac{\dot{p}}{p} = \gamma\left(\frac{\varepsilon}{\varepsilon-1}\right) p - (\rho + \gamma)$$

$$\frac{\dot{p}}{p} = \gamma\left(\frac{\varepsilon}{\varepsilon-1}\right) \cdot \left[p - p^*\right]$$

$$\rightarrow \boxed{\dot{p} = \gamma\left(\frac{\varepsilon}{\varepsilon-1}\right) p \left(p - p^*\right)}$$

Steady State: $\dot{p} = 0$

$$\rho + \gamma(1-p) = \frac{\gamma p}{\varepsilon - 1} \rightarrow \rho + \gamma = \gamma p \cdot \left(\frac{\varepsilon}{\varepsilon - 1}\right)$$

$$\rightarrow p^* = \left(\frac{\rho + \gamma}{\gamma}\right)\left(\frac{\varepsilon - 1}{\varepsilon}\right)$$

$$R^* = 1 - p^* \rightarrow g^* = \gamma R^* = \gamma(1 - p^*) = \gamma - \gamma p^*$$

$$g^* = \gamma - (\rho + \gamma)\left(\frac{\varepsilon - 1}{\varepsilon}\right) = \boxed{\frac{\gamma - (\varepsilon - 1)\rho}{\varepsilon} = g^*}$$

$$\boxed{\gamma > (\varepsilon - 1)\rho} \iff \begin{array}{l} g^* > 0 \\ R^* > 0 \\ p^* < 1 \end{array}$$

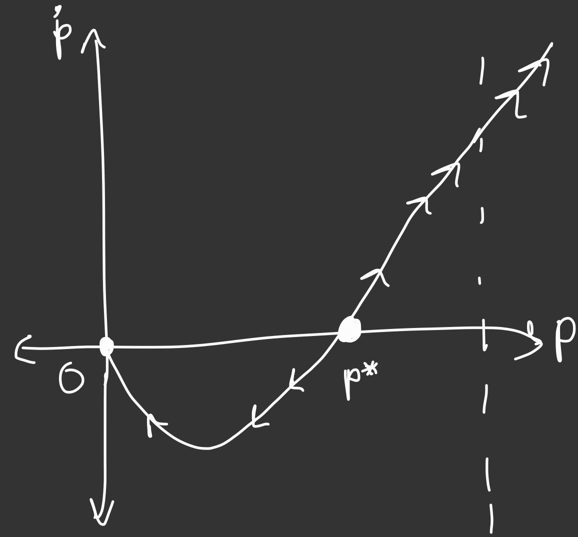
$1 = P + R$

$$\boxed{\dot{P} = \gamma\left(\frac{\varepsilon}{\varepsilon-1}\right)P\left(P-P^*\right)}$$

$$\to \quad g_P = \frac{\dot{P}}{P} = \gamma\left(\frac{\varepsilon}{\varepsilon-1}\right)\left(P-P^*\right)$$

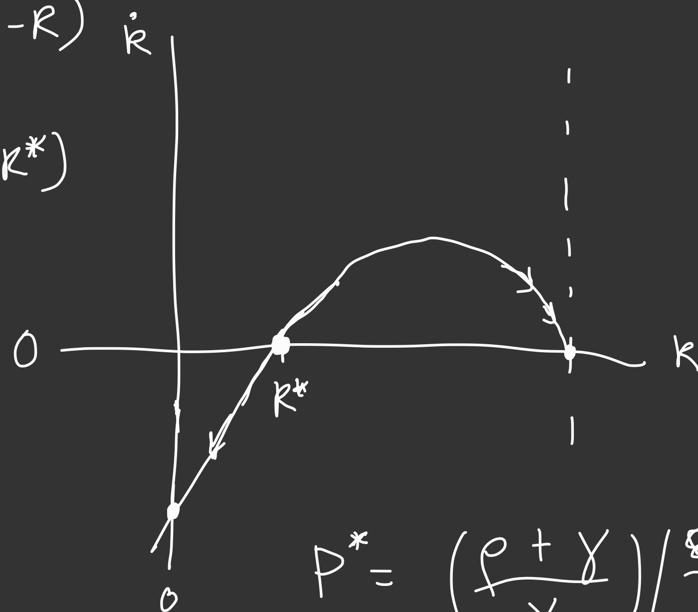
$$-\dot{R} = \gamma\left(\frac{\varepsilon}{\varepsilon-1}\right)(1-R)(1-R-P^*)$$

$$= \gamma\left(\frac{\varepsilon}{\varepsilon-1}\right)(1-R)(R^*-R)$$



$$\to \quad \dot{R} = -\gamma\left(\frac{\varepsilon}{\varepsilon-1}\right)(1-R)(R^*-R)$$

$$\dot{R} = -\gamma\left(\frac{\varepsilon}{\varepsilon-1}\right)(R-1)(R-R^*)$$



$$y = P \cdot N^{\frac{1}{\varepsilon-1}}$$

$R \to 1 \quad y?$ $\qquad g = \gamma R = \gamma$

$$g_y = g_P + \frac{g}{\varepsilon-1}$$

$$= -(\rho+\gamma) + \frac{\gamma}{\varepsilon-1}$$

$$= -\left[\rho + \gamma\left(\frac{2-\varepsilon}{\varepsilon}\right)\right]$$

$$P^* = \left(\frac{\rho+\gamma}{\gamma}\right) \Big/ \left(\frac{\varepsilon-1}{\varepsilon}\right) < 1$$

$$g_P = -\gamma\left(\frac{\varepsilon}{\varepsilon-1}\right)P^*$$

$$= -(\rho+\gamma)$$

$$\Rightarrow \quad \rho+\gamma < \varepsilon\left(\frac{\gamma}{\varepsilon-1}\right)$$

Equilibrium $\quad \dot{N} = \gamma N R \qquad 1 = P + R$

$$MB = \gamma N v = w = MC$$

$$w_R = w_P$$

$$v = \frac{\pi}{r - g_v}$$

$$\pi = \frac{wP}{(\varepsilon-1)N}$$

$$\gamma N \cdot \frac{\pi}{r - g_v} = w$$

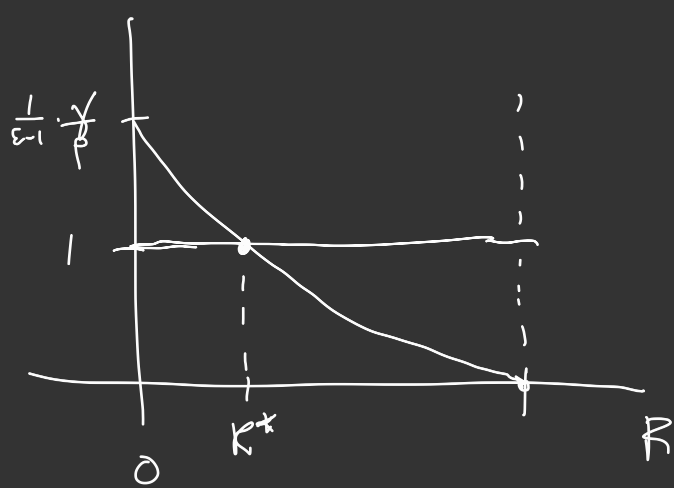
$$r - g_v = \rho + g + g_P = \rho + g$$

$$\to \gamma \cancel{N} \cdot \frac{1}{r-g_v} \cdot \frac{\cancel{w}P}{(\varepsilon-1)\cancel{N}} = \cancel{w}$$

$$\frac{1}{\varepsilon-1} \frac{\gamma P}{r - g_v} = 1$$

$$\to \frac{1}{\varepsilon-1} \cdot \frac{\gamma P}{\rho + g} = 1$$

$$\to \frac{1}{\varepsilon-1} \frac{\overbrace{\gamma(1-R)}^{\text{PMB}}}{\underbrace{\rho + \gamma R}_{}} = \overbrace{1}^{\text{PMC}}$$



Social Planner

N - state

l_i, R - choice

$$1 = \int_0^N l_i \, di + R$$

$$l_i = \frac{P}{N}$$

$$C = y = P \cdot N^{\frac{1}{\varepsilon-1}}$$

$$\dot{N} = \gamma N R$$

$$C = (1-R) \cdot N^{\frac{1}{\varepsilon-1}}$$

$$H = u\left((1-R)N^{\frac{1}{\varepsilon-1}}\right) + \mu \cdot \gamma N R$$

$$0 = H_R = -u'(C) \cdot N^{\frac{1}{\varepsilon-1}} + \mu \gamma N$$

$$\boxed{u'(C) N^{\frac{1}{\varepsilon-1}} = \mu \gamma N}$$

$$\rho\mu - \dot{\mu} = H_N = u'(c)(1-R)\left(\frac{1}{\varepsilon-1}\right)\frac{N^{\frac{1}{\varepsilon-1}}}{N} + \mu\gamma R$$

$$\rho\mu - \dot{\mu} = \mu\gamma(1-R)\left(\frac{1}{\varepsilon-1}\right) + \mu\gamma R$$

$$\rho - \frac{\dot{\mu}}{\mu} = \gamma\cdot\left[\frac{1-R}{\varepsilon-1} + R\right] \qquad \overset{\circ}{\underset{\text{"}}{y}} = p\cdot N^{\frac{1}{\varepsilon-1}}$$

$$\rightarrow \quad \rho + \cancel{g} + g_p = \frac{\gamma}{\varepsilon-1}(1-R) + \cancel{\gamma R}$$

$$\rho + g_p = \frac{\gamma}{\varepsilon-1}(1-R)$$

$$\rho - \frac{\dot{R}}{1-R} = \frac{\gamma}{\varepsilon-1}(1-R)$$

$$\boxed{u'(c)N^{\frac{1}{\varepsilon-1}} = \mu\gamma N}$$

$$\downarrow$$

$$-\varepsilon_{u'}\cdot g_c + \frac{g}{\varepsilon-1} = \frac{\dot{\mu}}{\mu} + g$$

$$-\theta g_c + \frac{g}{\varepsilon-1} = \frac{\dot{\mu}}{\mu} + g$$

$$\theta = 1 \quad (u = \log)$$

$$-\frac{\dot{\mu}}{\mu} = g - \frac{g}{\varepsilon-1} + g_c$$

$$= g - \frac{g}{\varepsilon-1} + \left(g_p + \frac{g}{\varepsilon-1}\right)$$

$$-\frac{\dot{\mu}}{\mu} = g + g_p$$

$$\boxed{\hat{R} = 1 - \frac{\rho}{\gamma}(\varepsilon-1)}$$

$$\frac{\dot{R}}{1-R} = \rho - \frac{\gamma}{\varepsilon-1}(1-R) = \rho - \frac{\gamma}{\varepsilon-1} + \frac{\gamma}{\varepsilon-1}R$$

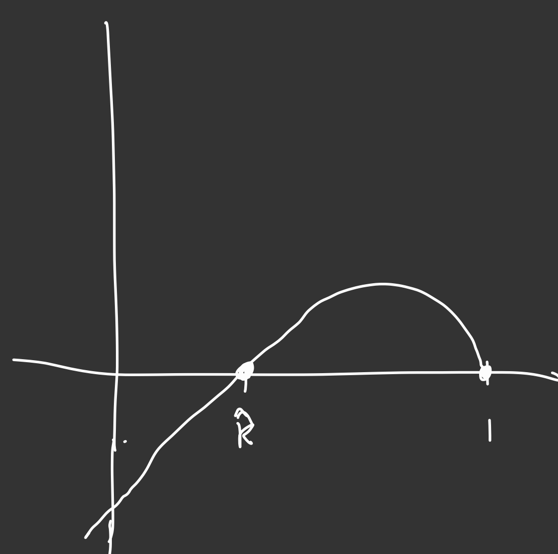
$$= \frac{\gamma}{\varepsilon-1}\cdot\left[\frac{\rho}{\gamma}(\varepsilon-1) - 1 + R\right] = \frac{\gamma}{\varepsilon-1}\Big[R - \underbrace{\left(1 - \frac{\rho}{\gamma}(\varepsilon-1)\right)}_{\hat{R}}\Big]$$

$$\dot{R} = \frac{\gamma}{\varepsilon-1}(1-R)(R-\hat{R})$$

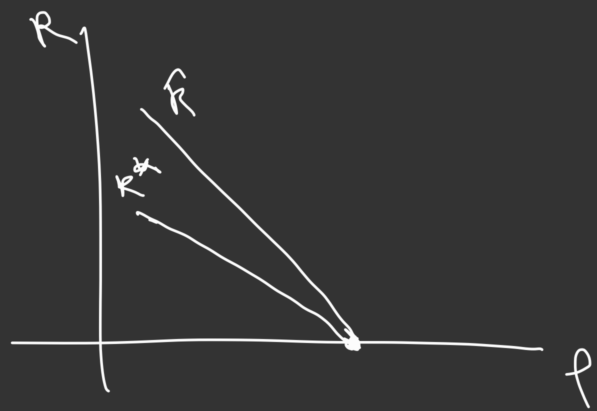
$$\rightarrow \quad \dot{R} = -\frac{\gamma}{\varepsilon-1}(R-1)(R-\hat{R})$$

$$\rightarrow \quad R = \hat{R}$$

$$\hat{R} = 1 - \frac{\rho}{\gamma}(\varepsilon-1)$$



$$\hat{R} = \frac{\gamma - (\varepsilon-1)\rho}{\gamma} > R^* = \frac{\gamma - (\varepsilon-1)\rho}{\varepsilon\gamma} \qquad g = \gamma R$$



$\dot{N} = \gamma N R$ ←

→ $\dot{N} v = w R$ ←

→ $\gamma N R v = w R$

→ $\boxed{\gamma N v = w}$

$[\gamma N \cdot v = w$

Limit pricing

$y_i = q_i l_i$

$l_i = \frac{y_i}{q_i} \to w l_i = \frac{w y_i}{q_i} \to MC_i = \frac{w}{q_i}$

$MC_{-i} = \frac{w}{q_{-i}} = \frac{w}{q_i/\lambda} = \frac{\lambda}{q_i} w$

$p_i^L = MC_{-i} = \frac{\lambda w}{q_i}$

↓

$\varepsilon = 1 \to p_i y_i = y \to$

$\to p_i = \frac{y}{y_i}$

$y_i = \frac{y}{p_i}$

$y_i^L = \frac{y}{p_i^L} = y \frac{q_i}{\lambda w}$

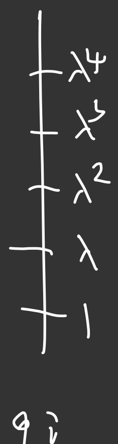
$\to \pi_i = \left(1 - \frac{1}{\lambda}\right) y$

$y_i = q_i l_i \to MC_i = \frac{w}{q_i}$

$q_{-i} = \frac{q_i}{\lambda} \to MC_{-i} = \frac{w}{q_{-i}} = \frac{\lambda w}{q_i}$

$p_i^L = MC_{-i} = \frac{\lambda w}{q_i}$

$\varepsilon = 1$ $\quad y = p_i y_i$



$\lambda > 1$ $\quad p_i = p_i^* = \dfrac{\lambda w}{q_i}$

$\rightarrow y_i = y \cdot \dfrac{q_i}{\lambda w}$

$\rightarrow l_i = \dfrac{y_i}{q_i} = \dfrac{y}{\lambda w}$

$\rightarrow w l_i = \dfrac{y}{\lambda} \rightarrow \dfrac{w l_i}{y} = \dfrac{1}{\lambda}$

$P = \int_0^1 l_i di \rightarrow \dfrac{wP}{y} = \dfrac{1}{\lambda}$

$\varepsilon = 1 \rightarrow p_i = \dfrac{y}{y_i} \quad y_i = \dfrac{y}{p_i}$

$\rightarrow p_i y_i = y$

$\pi_i = p_i y_i - w l_i$

$= y - \dfrac{y}{\lambda}$

$\boxed{\pi_i = (1 - \lambda^{-1}) \cdot y}$

$$y = \exp\left(\int_0^1 \log(y_i) di\right) = \frac{y}{\lambda w} \underbrace{\exp\left(\int_0^1 \log(q_i) di\right)}_{Q} = \frac{yQ}{\lambda w}$$

$$\rightarrow 1 = \frac{Q}{\lambda w} \rightarrow \boxed{w = \frac{Q}{\lambda}}$$

$$g \equiv \frac{\dot{Q}}{Q} = g_Q$$

$$P = \frac{y}{\lambda w} = \frac{y}{Q} \rightarrow \boxed{y = PQ}$$

$$y = \exp\left(\int_0^1 \log(q_i l_i) di\right) = \exp\left(\int_0^1 \log(q_i) di\right) \exp\left(\int_0^1 \log(l_i) di\right)$$

Value Function

$\Delta \tau$

τ – innov. rate

$\rightarrow x(t+\Delta) - x(t) \sim \text{Poisson}(\Delta \tau)$

$P(x > 0 | \mu)$

$= 1 - P(x = 0 | \mu) = 1 - e^{-\mu}$

$f(x|\mu) = \dfrac{\mu^x e^{-\mu}}{x!}$

$\mathbb{E}[x] = \mu$

$\text{Var}(x) = \mu$

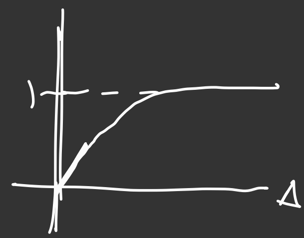
$\log f(x|\mu) = x \log(\mu) - \mu - \log(x!)$

$\dfrac{\partial \log f(x|\mu)}{\partial \mu} = \dfrac{x}{\mu} - 1 = \dfrac{x - \mu}{\mu}$

$\dfrac{x - \mu}{\sigma}$

$$\Pr(x>0 \mid \mu) = 1-\exp(-\mu)$$

$$\mu = \Delta\tau$$



$$\Pr(x>0 \mid \Delta\tau) = 1-\exp(-\Delta\tau) \approx \Delta\tau$$

$$V(t) = \Delta\pi(t) + \left[\Delta\tau \cdot 0 + (1-\Delta\tau)\cdot V(t+\Delta)\right]\exp(-\Delta r)$$

$$V(t) \approx \Delta\pi(t) + (1-\Delta r)(1-\Delta\tau)\, V(t+\Delta)$$

$$V(t) \approx \Delta\pi(t) + (1-\Delta(r+\tau))\, V(t+\Delta)$$

$$\Delta(r+\tau)V(t+\Delta) = \Delta\pi(t) + V(t+\Delta) - V(t)$$

$$(r+\tau)\, V(t+\Delta) = \pi(t) + \frac{V(t+\Delta)-V(t)}{\Delta}$$

$$(r+\tau)V = \pi + \dot{V} \rightarrow \boxed{(r+\tau)V - \dot{V} = \pi}$$

$$r+\tau - \frac{\dot{V}}{V} = \frac{\pi}{V} \rightarrow \boxed{V = \frac{\pi}{r+\tau-g_V}}$$

$$g_V = g_\pi = g_y = g$$

$$\rightarrow \tau = \gamma R$$

$$\rightarrow \gamma \cdot V = w$$

$$g = \log(\lambda)\cdot\tau$$

Value Function stuff

$$V_0 = \frac{\pi_0}{r}$$

$V_0 = 0$ old : $(r+\tau)v - \dot{v} = \pi \rightarrow rv - \dot{v} = \pi + \tau \cdot (0 - v)$

V_0 new : $v(t) = \Delta\pi(t) + \exp(-\Delta r) \cdot \left[\Delta\tau \cdot V_0 + (1-\Delta\tau) v(t+\Delta)\right]$

$$v(t) = \Delta\pi(t) + (1-\Delta r)\left[v(t+\Delta) + \Delta\tau \cdot \left(V_0 - v(t+\Delta)\right)\right]$$

$$\Delta r v(t+\Delta) + \left[v(t) - v(t+\Delta)\right] = \Delta\pi(t) + (1-\Delta r)\Delta\tau\left[V_0 - v(t+\Delta)\right]$$

$$rv(t+\Delta) + \frac{v(t) - v(t+\Delta)}{\Delta} = \pi(t) + (1-\Delta r)\tau\left[v_0 - v(t+\Delta)\right]$$

$\lim_{\Delta \to 0}$ $$rv - \dot{v} = \pi + \tau(v_0 - v) + \mu(v_1 - v)$$

Schumpeter |

$$\pi = (1-\lambda^{-1})y = (1-\lambda^{-1})QP$$

$$\rightarrow wP + \pi = y$$

$$w = \frac{Q}{\lambda}$$

$$y = PQ$$

$$V = \frac{\pi}{r+\tau-g_v}$$

$$V = \frac{\pi}{\rho+\tau}$$

$\phi < 1$: $g^* = \frac{n\eta}{1-\phi}$

$\phi = 1$: $g = \gamma R^\eta$

$\phi >$: singularity

$\left[\dot{A} = \gamma A^\phi R^\eta\right.$

Free Entry | $\rightarrow \tau = \gamma R$

$$\boxed{\gamma v = w}$$

$r = \rho + g_c = \rho + g$

$g_v = g_w = g_Q \equiv g$

$r + \tau - g = \rho + \tau$

$g = \frac{\dot{Q}}{Q} = \log(\lambda)\tau$

$\rightarrow \dot{Q} = \log(\lambda) Q \tau = \gamma \log(\lambda) Q^{\phi=1} R^{\eta=1} = \dot{Q}$

$Q \leftrightarrow A$

$\dot{N} = \gamma N^{\phi=1} R^{\eta=1}$

Free Entry

$$\gamma V = w$$

$$\frac{\gamma \pi}{\rho + \tau} = \frac{Q}{\lambda} \rightarrow \frac{\gamma(1-\lambda^{-1})y}{\rho+\tau} = \frac{Q}{\lambda}$$

$$\rightarrow \frac{\gamma(1-\lambda^{-1})QP}{\rho + \gamma R} = \frac{Q}{\lambda} \rightarrow \gamma(\lambda-1)P = \rho + \gamma R$$

$$\rightarrow \gamma(\lambda-1)(1-R) = \rho + \gamma R$$

$$\rightarrow R^* = \frac{\gamma(\lambda-1)-\rho}{\gamma\lambda} = (1-\lambda^{-1}) - \frac{\rho}{\lambda\gamma}$$

$$\tau^* = \gamma R^* = \frac{\gamma(\lambda-1)-\rho}{\lambda} \rightarrow g^* = \frac{\log(\lambda)}{\lambda} \cdot [\gamma(\lambda-1)-\rho]$$

$$\log(\lambda)\cdot\left[\gamma\left(\frac{\lambda-1}{\lambda}\right) - \frac{\rho}{\lambda}\right]$$

$$\log(\lambda)\cdot\gamma$$

$$g = \frac{\dot{Q}}{Q} = \frac{d\log(Q)}{dt} \qquad \log(Q) = \int_0^1 \log(q_i)\, di$$

$$Q = \left[\int_0^1 q_i^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$Q^{\frac{\varepsilon-1}{\varepsilon}} = \int_0^1 q_i^{\frac{\varepsilon-1}{\varepsilon}} di$$

$$\Delta\tau$$

$$\log Q(t+\Delta) = \int_0^1 \log q_i(t+\Delta)\, di$$

$$= \int_0^1 \left[\Delta\tau\cdot\log(\lambda q_i(t)) + (1-\Delta\tau)\cdot\log(q_i(t))\right] di$$

$$= \int_0^1 \left[\Delta\tau\cdot\log(\lambda) + \log(q_i(t))\right] di$$

$$\log Q(t+\Delta) = \Delta\tau\log(\lambda) + \log Q(t)$$

$$\boxed{g = \log(\lambda)\tau}$$

$$\rightarrow \frac{\log Q(t+\Delta) - \log Q(t)}{\Delta} = \log(\lambda)\cdot\tau$$

$$\rightarrow \frac{d\log Q}{dt} = \log(\lambda)\tau$$

Social Planner

$y_i = q_i l_i$

$\log(y) = \int_0^1 \log(y_i)\,di$

$\log(y) = \int_0^1 \log(q_i l_i)\,di$

$\uparrow$

$u = \log(y) = \int_0^1 \log(q_i)\,di + \int_0^1 \log(l_i)\,di \rightarrow \log(y) = \log(Q) + \int_0^1 \log(l_i)\,l_i$

$Q \rightarrow g = \log(\lambda)\cdot\tau$

$\vec{q}$? $f(q)\ /\ \hat{f}(\hat{q})$ $\quad \hat{q}_i = q_i/Q$

$q_i \begin{cases} \nearrow \Delta\tau \rightarrow \lambda \\ \searrow 1-\Delta\tau \rightarrow 1 \end{cases}$

$\hat{q}_i \begin{cases} \nearrow \Delta\tau \quad \frac{\lambda}{1+\Delta g} \\ \searrow 1-\Delta\tau \quad \frac{1}{1+\Delta g} \end{cases}$

$\rightarrow \log \hat{q}_i \begin{cases} \nearrow \Delta\tau \quad \log(\lambda) - \log(1+\Delta g) > 0 \\ \searrow 1-\Delta\tau \quad -\log(1+\Delta g) \end{cases}$

$\frac{\partial u}{\partial l_i} = \frac{1}{l_i}$

$\Rightarrow y = Q\cdot\exp\left(\int_0^1 \log(l_i)\,di\right)$

$\frac{\partial y}{\partial l_i} = \frac{Q}{l_i}$

Hamilton-ian

$u(c) = \log(c) = \log(y)$

$y = QP = Q(1-R)$

$H = u(c) + \mu\cdot\dot{Q}$

$H = \log(Q(1-R)) + \mu\log(\lambda)\gamma RQ$

Q - state

R - control

$\dot{Q} = Q\cdot\log(\lambda)\cdot\tau = \log(\lambda)\gamma RQ$

$\dot{Q}/Q = g$

$g = \log(\lambda)\tau$

$\tau = \gamma R$

$\rightarrow H = \log(Q) + \log(1-R) + \mu\log(\lambda)\gamma RQ$

① $0 = H_R = \frac{-1}{1-R} + \mu\log(\lambda)\gamma Q \rightarrow \frac{1}{1-R} = \mu\log(\lambda)\gamma Q$

② $\rho\mu - \dot{\mu} = H_Q = \frac{1}{Q} + \mu\log(\lambda)\gamma R$ $\qquad \rho\mu - \dot{\mu} = \mu\log(\lambda)\gamma$

$\rho\mu - \dot{\mu} = \mu\log(\lambda)\gamma(1-R) + \mu\log(\lambda)\gamma R = \mu\log(\lambda)\gamma$

$$\frac{1}{1-R} = \mu\log(\lambda)\gamma Q \;\to\; \frac{+\dot R}{(1-R)^2}\Big/\left(\frac{1}{1-R}\right) = \frac{\dot R}{1-R} = \frac{\dot\mu}{\mu} + g = \frac{\dot\mu}{\mu} + \log(\lambda)\gamma R$$

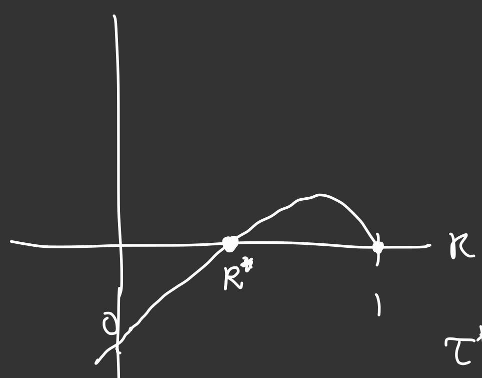
$$\rho\mu - \dot\mu = \mu\log(\lambda)\gamma \;\to\; \frac{\dot\mu}{\mu} = \rho - \log(\lambda)\gamma$$

$$\to\; \frac{\dot R}{1-R} = \rho - \log(\lambda)\gamma + \log(\lambda)\gamma R = \rho - \log(\lambda)\gamma(1-R)$$

$$\to\; \frac{\dot R}{1-R} = \log(\lambda)\gamma\cdot\left[\frac{\rho}{\log(\lambda)\gamma} - 1 + R\right]$$

$$\to\; \dot R = -(R-1)\left(R - \left(1 - \frac{\rho}{\log(\lambda)\gamma}\right)\right)\log(\lambda)\gamma$$

$$\to\; \dot R = -\log(\lambda)\gamma\,(R-1)(R-\hat R) \qquad \boxed{\hat R = 1 - \frac{\rho}{\log(\lambda)\gamma}}$$

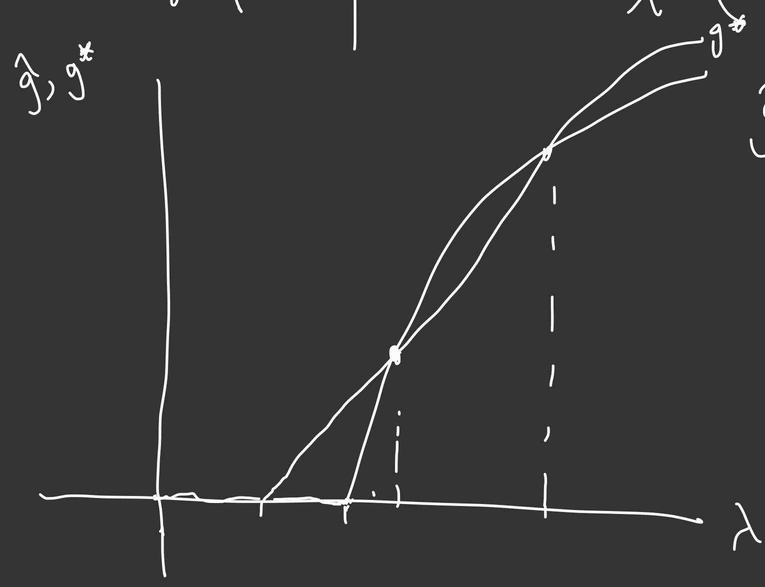


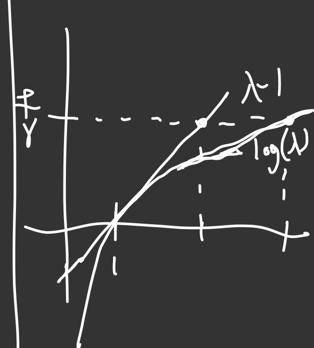
$$\hat\tau = \gamma\hat R = \gamma - \frac{\rho}{\log(\lambda)}$$

$$\hat g = \log(\lambda)\hat\tau = \log(\lambda)\gamma - \rho$$

$$\tau^* = \frac{\gamma(\lambda-1)-\rho}{\lambda}$$

$$\hat g = \log(\lambda)\gamma - \rho \quad\Big|\quad g^* = \frac{\log(\lambda)}{\lambda}\big((\lambda-1)\gamma - \rho\big) \quad\Big|\quad \log(\lambda) \simeq \lambda - 1$$





Schumpeter $\tilde{\lambda} = \log(\lambda)$

Equil.

$$R^* = \frac{1}{\lambda}\left[(\lambda-1) - \rho/\gamma\right]$$

$$g^* = \frac{\tilde{\lambda}}{\lambda}\left[(\lambda-1)\gamma - \rho\right]$$

Soc. plan.

$$\hat{R} = 1 - \frac{\rho}{\tilde{\lambda}\gamma}$$

$$\hat{g} = \tilde{\lambda}\gamma - \rho$$

$y = QP$

$u(c) = \log(Qp)$
$= \log(Q(1-R))$

discount ρ

PMB = PMC

SMB = SMC choosing R

$$\gamma V = w \qquad \pi = (1-\lambda^{-1})y$$

$$\rightarrow \frac{\gamma\tilde{\lambda}}{\rho} = \frac{1}{1-R}$$

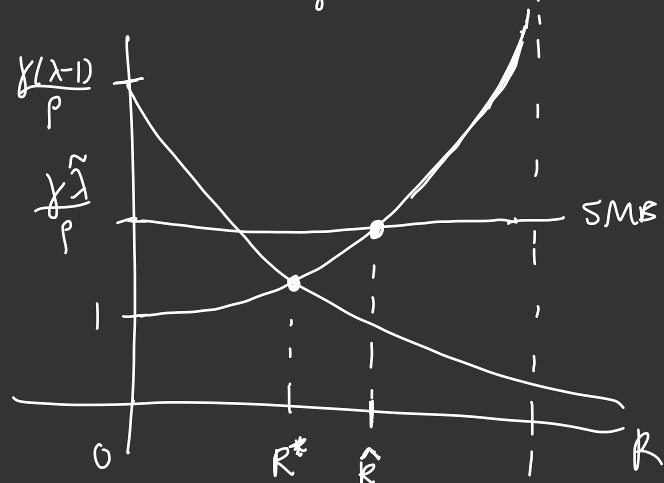
$$\rightarrow \frac{\gamma\pi}{\rho+\tau} = w \qquad wP = \lambda^{-1}y$$

$$\rightarrow \hat{R} = 1 - \frac{\rho}{\gamma\tilde{\lambda}}$$

$$\frac{\gamma(1-\lambda^{-1})y}{\rho+\gamma R} = \frac{\lambda^{-1}y}{P}$$

$$\frac{\gamma(1-\lambda^{-1})}{\rho+\gamma R} = \frac{\lambda^{-1}}{1-R}$$

$$\widetilde{PMB} = \frac{\gamma(\lambda-1)}{\rho+\gamma R} = \frac{1}{1-R} = \widetilde{PMC}$$



Numbers

$\rho = 3\%$ $\rightarrow \frac{\pi}{y} = 1 - \lambda^{-1}$

$\lambda - 1 = 15\%$ $\rightarrow \frac{\pi}{y} - \frac{wR}{y} = (1-\lambda^{-1}) - \lambda^{-1}\cdot\frac{R}{1-R}$

$\gamma = 1.3$

$\rightarrow g^* = 2\%$ $\hat{g} = 15\%$

$$y_i = \frac{\alpha_i q_i}{1+\lambda} \cdot \frac{y}{w} = \alpha_i q_i \cdot p$$

$$y =_\lambda Q P \qquad \log(Q) = \int_0^1 \alpha_i \log(q_i) di$$

$$\log Q(t+\Delta) = \int_0^1 \alpha_i \log(q_i(t+\Delta)) di$$

$$= \int_0^1 \alpha_i \cdot \left[\Delta \tau_i \cdot \log(\lambda q_i(t)) + (1-\Delta \tau_i) \log(q_i(t))\right] di$$

$$= \Delta \int_0^1 \alpha_i \tau_i \log(\lambda) di + \int_0^1 \alpha_i \log(q_i(t)) di$$

$$= \Delta \log(\lambda) \int_0^1 \alpha_i \tau_i di + \log(Q(t))$$

$$g = \lim_{\Delta \to 0} \frac{\log Q(t+\Delta) - \log Q(t)}{\Delta} = \log(\lambda) \int_0^1 \alpha_i \tau_i di$$

$$p_i = \frac{w}{q_i} \qquad p_i y_i = y$$

$$\to y_i = \frac{y}{p_i} = \frac{y q_i}{w}$$

$$\to l_i^c = \frac{y}{w} \qquad l_i^m = \frac{1}{\lambda} \cdot \frac{y}{w}$$

$$1-\mu \qquad \mu$$

$$l_i^c = \frac{P}{1-\mu\left(\frac{\lambda-1}{\lambda}\right)}$$

$$P = \mu \cdot \frac{1}{\lambda} \cdot \frac{y}{w} + (1-\mu) \frac{y}{w}$$

$$P = \frac{y}{w}\left[\frac{\mu}{\lambda} + 1 - \mu\right] = \frac{y}{w}\left[1-\mu\left(\frac{\lambda-1}{\lambda}\right)\right]$$

$$\to \boxed{w = \frac{y}{P}\left[1-\mu\left(\frac{\lambda-1}{\lambda}\right)\right]}$$