Chapter 5
Endogenous Growth

Econ 3070: Macroeconomics 2.0

University of Pittsburgh, 2020

First Steps

We can think about a "semi-endogenous" growth model with the law of motion

$$\dot{A} = \gamma A^{\phi} R^{\eta}$$

where R is the amount of resources devoted to research (usually labor)

In terms of growth rates this implies

$$g \equiv \frac{\dot{A}}{A} = \frac{\gamma R^{\eta}}{A^{1-\phi}}$$

so for $\phi \in (0, 1)$, higher A makes growth proportionately harder

Steady State

In steady state, constant g means the numerator and denominator grow at the same rate

$$\begin{aligned} \eta g_R &= (1-\phi)g \\ \Rightarrow \quad g &= \frac{\eta}{1-\phi} g_R \end{aligned}$$

If researchers constitute a fixed fraction of the population ($R = sL$) and population grows at rate n

$$g = \frac{\eta}{1-\phi} n$$

Notice this doesn't depend on s!

Jones Taxonomy

There are three major cases for the value of ϕ

- $\phi < 1$: fixed long-run growth rate
- $\phi = 1$: growth rate depends on s
- $\phi > 1$: wild stuff (more later)

Most growth models, in the aggregate, reduce to some form like this

- many will fall in the "kife-edge" case where $\phi = 1$

Knife Edge

In the specific case where $\phi = 1$, we find something qualitatively different

$$g = \gamma R^\eta = \gamma (sL)^\eta$$

Notice that now if we have population growth, the *growth rate* grows exponentially!

This forces us to choose between a formulation with $\phi = 1$ with no population growth or a $\phi < 1$ formulation (sometimes trickier)

Singularity

We'll usually avoid the $\phi > 1$ case because it leads to an singularity as

$$\frac{\dot{g}}{g} = \eta n - (1 - \phi) g$$

To see this, suppose that $n = 0$, then we have

$$\frac{\dot{g}}{g} = (\phi - 1) g \quad \Rightarrow \quad \dot{g} = (\phi - 1) g^2$$

This has a solution reaching an infinite value in finite time when $\phi > 1$

$$g(t) = \frac{g_0}{1 - g_0(\phi - 1)t}$$

Fully Endogenous

Expanding Varieties

In general, there are two classes of model depending on where growth comes from

- expanding varieties: growth from the creation of new products (Romer)
- quality ladder: growth from better/cheaper products (Schumpteter)

These often have similar welfare implications, but the creative destruction in Schumpeterian models is distinctive

Goods Aggregator (🐊)

One of the core objects in many growth models is how we map from individual varieties to total output

Let's go with a CES aggregator with parameter ε in this case

$$y = \left[\int_0^N y_i^{\frac{\varepsilon-1}{\varepsilon}}\, di\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

A price-taking firm operating this technology will maximize

$$\Pi = p\, y - \int_0^N p_i y_i di$$

Demand Function

Optimality dictates that the firm should choose y_i optimally for all i

$$p_i = \frac{\partial y}{\partial y_i} = \left(\frac{y_i}{y}\right)^{-1/\varepsilon}$$

It is without loss of generality to assume that $p_y = 1$ at all times t

Any dynamics can be accounted for with a time varying interest rate

Intermediate Firms

The main actors are intermediate firms that choose how much of y_i to produce given this demand function

Each good i is produced by a different monopolistic firm i (say the firm has an infinite length patent on the product)

They can produce y_i linearly with labor ℓ_i (at wage w), so that $y_i = \ell_i$

Profit Maximization

For general $p(y)$, a firm's quantity choice problem is

$$\begin{aligned} \pi &= p(y)y - w\ell = (p(y) - w)y \\ \Rightarrow \quad & (p(y) - w) + p'(y)y = 0 \end{aligned}$$

We can rearrange this into a markup form

$$\begin{aligned} \frac{p-w}{p} &= -\frac{p'(y)y}{p(y)} \equiv \frac{1}{\varepsilon} \\ \Rightarrow \quad \frac{p}{w} &= \frac{\varepsilon}{\varepsilon - 1} \end{aligned}$$

Firm Profit

By symmetry, firms will all choose the same y_i and hence ℓ_i, so we have

$$P = \int_0^N \ell_i di = N\ell_i \quad \Rightarrow \quad \ell_i = \frac{P}{N}$$

We can also derive the profit margin equation

$$\frac{\pi_i}{w\ell_i} = \frac{p_i y_i - w\ell_i}{w\ell_i} = \frac{p_i}{w} - 1 = \frac{1}{\varepsilon - 1}$$

Combining these yields the absolute profit level

$$\pi_i = \frac{wP}{(\varepsilon - 1)N}$$

Firm Value

Now we map profit to value. For discount rate r, the value should satisfy

$$rv - \dot{v} = \pi \quad \Rightarrow \quad v = \frac{\pi}{r - g_v}$$

If one researcher discovers new products at rate γ, our ideas production function is

$$\dot{N} = \gamma N R$$

We should have the follow zero profit condition condition

$$0 = \dot{N} v - wR = R\left[\gamma N v - w\right]$$

Free Entry

Thus either there is no entry $R = 0$ or we should satisfy the free entry condition (then lots of algebra)

$$\begin{aligned}
\gamma N v &= w \\
\gamma N \frac{\pi}{r - g_v} &= w \\
\frac{\gamma N}{r - g_v} \frac{wP}{(\varepsilon - 1)N} &= w \\
\frac{\gamma}{r - g_v} \frac{P}{\varepsilon - 1} &= 1
\end{aligned}$$

Rearranging we find

$$r - g_v = \frac{\gamma P}{\varepsilon - 1}$$

Aggregate Output

Since $y_i = \ell_i = P/N$, we can use the aggregator to get

$$y = \left[\int_0^N \left(\frac{P}{N}\right)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} = \frac{P}{N} N^{\frac{\varepsilon}{\varepsilon-1}} = PN^{\frac{1}{\varepsilon-1}}$$

Thus the growth rate of output (and hence consumption) satisfies

$$g_c = g_y = g_P + \frac{g}{\varepsilon - 1}$$

Additionally, our Euler equation (using log utilty) still applies

$$r = \rho + g_c$$

Wage Rates

What about the wage? That is determined by the demand for labor

$$\begin{aligned} \left(\frac{y_i}{y}\right)^{-1/\varepsilon} &= p = \left(\frac{\varepsilon - 1}{\varepsilon}\right) w \\ \Rightarrow \quad y_i &= \left[\frac{\varepsilon - 1}{\varepsilon}\frac{1}{w}\right]^{\varepsilon} y \\ \Rightarrow \quad \ell_i &= P\left[\frac{\varepsilon - 1}{\varepsilon}\frac{1}{w}\right]^{\varepsilon} N^{\frac{1}{\varepsilon-1}} \end{aligned}$$

Imposing labor market cleaning allows us to integrate and solve for w

$$w = \frac{\varepsilon - 1}{\varepsilon} N^{\frac{1}{\varepsilon-1}} \quad \Rightarrow \quad g_w = \frac{g}{\varepsilon - 1}$$

Growth Rates

The free entry condition in growth form ammounts to

$$g_v + g = g_w$$

Combing all of the growth rate results thus far we find

$$r - g_v = \rho + g_c - g_v = \rho + g + g_P$$

And so finally we arrive at the condition

$$\rho + g + g_P = \frac{\gamma P}{\varepsilon - 1}$$

Steady State

First let's find steady state by setting $g_P = 0$

Note that here $g = \gamma R$ and $P + R = 1$, meaning

$$\rho + \gamma R = \frac{\gamma(1 - R)}{\varepsilon - 1}$$

This yields a steady state value for research of

$$R^* = \frac{\gamma - (\varepsilon - 1)\rho}{\varepsilon\gamma} \quad \Rightarrow \quad g^* = \frac{\gamma - (\varepsilon - 1)\rho}{\varepsilon}$$

Notice that this may be positive or zero depending on parameters

Full Dynamics

Now let's allow for $g_P = \frac{\dot{P}}{P} = -\frac{\dot{R}}{1-R}$ so that

$$\rho + \gamma R - \frac{\dot{R}}{1-R} = \frac{\gamma(1-R)}{\varepsilon - 1}$$

This yields a law of motion equation for research R

$$\begin{aligned} \dot{R} &= (1-R)\left[\frac{\varepsilon\gamma R}{\varepsilon - 1} - \frac{\gamma - (\varepsilon - 1)\rho}{\varepsilon - 1}\right] \\ &= \gamma\left(\frac{\varepsilon}{\varepsilon - 1}\right)(1-R)(R - R^*) \end{aligned}$$

Single Outcome

Law of motion crosses zero at $R = R^*$ and $R = 1$

— The $R = R^*$ steady state is unstable
— The $R = 1$ steady state is stable

If we start at $R < R^*$ eventually R will go negative, violating feasibility

If we start at $R > R^*$ then R will converge to 1, meaning $P = 0$ and $y = c = 0$, violating transversality

Thus $R = R^*$ is the only feasible outcome and we start there from time zero