Chapter 5

Endogenous Growth

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First Steps

We can think about a "semi-endogenous" growth model with the law of motion

$$\dot{A} = \gamma A^{\phi} R^{\eta}$$

where R is the amount of resources devoted to research (usually labor)

In terms of growth rates this implies

$$g \equiv \frac{\dot{A}}{A} = \frac{\gamma R^{\eta}}{A^{1-\phi}}$$

so for $\phi \in (0, 1)$, higher A makes growth proportionately harder

Steady State

In steady state, constant g means the numerator and denominator grow at the same rate

$$\begin{aligned} \eta g_R &= (1-\phi) g \\ \Rightarrow \quad g &= \frac{\eta}{1-\phi} g_R \end{aligned}$$

If researchers constitute a fixed fraction of the population ($R = sL$) and population grows at rate n

$$g = \frac{\eta}{1-\phi} n$$

Notice this doesn't depend on s!

Jones Taxonomy

There are three major cases for the value of ϕ

- $\phi < 1$: fixed long-run growth rate
- $\phi = 1$: growth rate depends on s
- $\phi > 1$: wild stuff (more later)

Most growth models, in the aggregate, reduce to some form like this

- many will fall in the "kife-edge" case where $\phi = 1$

Knife Edge

In the specific case where $\phi = 1$, we find something qualitatively different

$$g = \gamma R^\eta = \gamma (sL)^\eta$$

Notice that now if we have population growth, the *growth rate* grows exponentially!

This forces us to choose between a formulation with $\phi = 1$ with no population growth or a $\phi < 1$ formulation (sometimes trickier)

Singularity

We'll usually avoid the $\phi > 1$ case because it leads to an singularity as

$$\frac{\dot{g}}{g} = \eta n - (1 - \phi) g$$

To see this, suppose that $n = 0$, then we have

$$\frac{\dot{g}}{g} = (\phi - 1) g \quad \Rightarrow \quad \dot{g} = (\phi - 1) g^2$$

This has a solution reaching an infinite value in finite time when $\phi > 1$

$$g(t) = \frac{g_0}{1 - g_0 (\phi - 1) t}$$

Fully Endogenous

Expanding Varieties

In general, there are two classes of model depending on where growth comes from

- expanding varieties: growth from the creation of new products (Romer)
- quality ladder: growth from better/cheaper products (Schumpteter)

These often have similar welfare implications, but the creative destruction in Schumpeterian models is distinctive

Goods Aggregator (🐊)

One of the core objects in many growth models is how we map from individual varieties to total output

Let's go with a CES aggregator with parameter ε in this case

$$y = \left[\int_0^N y_i^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

A price-taking firm operating this technology will maximize

$$\Pi = p\, y - \int_0^N p_i y_i di$$

Demand Function

Optimality dictates that the firm should choose y_i optimally for all i

$$p_i = \frac{\partial y}{\partial y_i} = \left(\frac{y_i}{y}\right)^{-1/\varepsilon}$$

It is without loss of generality to assume that $p_y = 1$ at all times t

Any dynamics can be accounted for with a time varying interest rate

Intermediate Firms

The main actors are intermediate firms that choose how much of y_i to produce given this demand function

Each good i is produced by a different monopolistic firm i (say the firm has an infinite length patent on the product)

They can produce y_i linearly with labor ℓ_i (at wage w), so that $y_i = \ell_i$

Profit Maximization

For general $p(y)$, a firm's quantity choice problem is

$$\begin{aligned} & \pi = p(y)y - w\ell = (p(y) - w)y \\ \Rightarrow \quad & (p(y) - w) + p'(y)y = 0 \end{aligned}$$

We can rearrange this into a markup form

$$\begin{aligned} & \frac{p-w}{p} = -\frac{p'(y)y}{p(y)} \equiv \frac{1}{\varepsilon} \\ \Rightarrow \quad & \frac{p}{w} = \frac{\varepsilon}{\varepsilon - 1} \end{aligned}$$

Firm Profit

By symmetry, firms will all choose the same y_i and hence ℓ_i, so we have

$$P = \int_0^N \ell_i di = N\ell_i \quad \Rightarrow \quad \ell_i = \frac{P}{N}$$

We can also derive the profit margin equation

$$\frac{\pi_i}{w\ell_i} = \frac{p_i y_i - w\ell_i}{w\ell_i} = \frac{p_i}{w} - 1 = \frac{1}{\varepsilon - 1}$$

Combining these yields the absolute profit level

$$\pi_i = \frac{wP}{(\varepsilon - 1)N}$$

Firm Value

Now we map profit to value. For discount rate r, the value should satisfy

$$rv - \dot{v} = \pi \quad \Rightarrow \quad v = \frac{\pi}{r - g_v}$$

If one researcher discovers new products at rate γ, our ideas production function is

$$\dot{N} = \gamma N R$$

We should have the follow zero profit condition condition

$$0 = \dot{N} v - wR = R\left[\gamma N v - w\right]$$

Free Entry

Thus either there is no entry $R = 0$ or we should satisfy the free entry condition (then lots of algebra)

$$\begin{aligned}
\gamma N v &= w \\
\gamma N \frac{\pi}{r - g_v} &= w \\
\frac{\gamma N}{r - g_v} \frac{wP}{(\varepsilon - 1)N} &= w \\
\frac{\gamma}{r - g_v} \frac{P}{\varepsilon - 1} &= 1
\end{aligned}$$

Rearranging we find

$$r - g_v = \frac{\gamma P}{\varepsilon - 1}$$

Aggregate Output

Since $y_i = \ell_i = P/N$, we can use the aggregator to get

$$y = \left[\int_0^N \left(\frac{P}{N}\right)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} = \frac{P}{N} N^{\frac{\varepsilon}{\varepsilon-1}} = PN^{\frac{1}{\varepsilon-1}}$$

Thus the growth rate of output (and hence consumption) satisfies

$$g_c = g_y = g_P + \frac{g}{\varepsilon - 1}$$

Additionally, our Euler equation (using log utilty) still applies

$$r = \rho + g_c$$

Wage Rates

What about the wage? That is determined by the demand for labor

$$\left(\frac{y_i}{y}\right)^{-1/\varepsilon} = p = \left(\frac{\varepsilon}{\varepsilon - 1}\right) w$$

$$\Rightarrow \quad y_i = \left[\frac{\varepsilon - 1}{\varepsilon}\frac{1}{w}\right]^{\varepsilon} y$$

$$\Rightarrow \quad \ell_i = P\left[\frac{\varepsilon - 1}{\varepsilon}\frac{1}{w}\right]^{\varepsilon} N^{\frac{1}{\varepsilon - 1}}$$

Imposing labor market cleaning allows us to integrate and solve for w

$$w = \frac{\varepsilon - 1}{\varepsilon} N^{\frac{1}{\varepsilon - 1}} \quad \Rightarrow \quad g_w = \frac{g}{\varepsilon - 1}$$

Growth Rates

The free entry condition in growth form ammounts to

$$g_v + g = g_w$$

Combing all of the growth rate results thus far we find

$$r - g_v = \rho + g_c - g_v = \rho + g + g_P$$

And so finally we arrive at the condition

$$\rho + g + g_P = \frac{\gamma P}{\varepsilon - 1}$$

Steady State

First let's find steady state by setting $g_P = 0$

Note that here $g = \gamma R$ and $P + R = 1$, meaning

$$\rho + \gamma R = \frac{\gamma(1 - R)}{\varepsilon - 1}$$

This yields a steady state value for research of

$$R^* = \frac{\gamma - (\varepsilon - 1)\rho}{\varepsilon\gamma} \quad \Rightarrow \quad g^* = \frac{\gamma - (\varepsilon - 1)\rho}{\varepsilon}$$

Notice that this may be positive or zero depending on parameters

Full Dynamics

Now let's allow for $g_P = \frac{\dot{P}}{P} = -\frac{\dot{R}}{1-R}$ so that

$$\rho + \gamma R - \frac{\dot{R}}{1-R} = \frac{\gamma(1-R)}{\varepsilon - 1}$$

This yields a law of motion equation for research R

$$\begin{aligned}\dot{R} &= (1-R)\left[\frac{\varepsilon\gamma R}{\varepsilon - 1} - \frac{\gamma - (\varepsilon - 1)\rho}{\varepsilon - 1}\right] \\ &= \gamma\left(\frac{\varepsilon}{\varepsilon - 1}\right)(1-R)(R - R^*)\end{aligned}$$

Single Outcome

Law of motion crosses zero at $R = R^*$ and $R = 1$

- The $R = R^*$ steady state is unstable
- The $R = 1$ steady state is stable

If we start at $R < R^*$ eventually R will go negative, violating feasibility

If we start at $R > R^*$ then R will converge to 1, meaning $P = 0$ and $y = c = 0$, violating transversality

Thus $R = R^*$ is the only feasible outcome and we start there from time zero

Social Planner

State and Choice

At any given time, the state of technology is characterized by the number of products N

The planner must choose how much labor to allocate to each product line ℓ_i as well as the amount of research labor R subject to

$$\int_0^N \ell_i di + R = 1$$

We know due to symmetry that we will end up with $\ell_i = L/N$, but let's show it formally

Hamiltonian

The Hamiltonian for the social planner's problem is then

$$H = u\left(\left[\int_0^N \ell_i^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}}\right) + \mu\gamma N\left(1 - \int_0^N \ell_i di\right)$$

The first order condition for ℓ_i yields

$$0 = H_{\ell_i} = u'(c)\left(\frac{y}{\ell_i}\right)^{\frac{1}{\varepsilon}} - \mu\gamma N$$

From this we can already see that ℓ_i will be constant across i, which then implies after some algebra that

$$u'(c)N^{\frac{1}{\varepsilon-1}} = \mu\gamma N$$

Optimality

Next the state evolution equation ends up being (Leibnitz rules!)

$$
\begin{aligned}
\rho\mu - \dot{\mu} = H_N &= u'(c)\left(\frac{\varepsilon}{\varepsilon-1}\right) y^{\frac{1}{\varepsilon}} \ell_N^{\frac{\varepsilon-1}{\varepsilon}} - \mu\gamma N \ell_N + \mu\gamma\left(1 - \int_0^N \ell_i di\right) \\
&= u'(c)\left(\frac{\varepsilon}{\varepsilon-1}\right)\frac{P}{N} N^{\frac{1}{\varepsilon-1}} - \mu\gamma P + \mu g
\end{aligned}
$$

Substituting the first order condition here substantially simplifies things

$$
\begin{aligned}
&\rho\mu - \dot{\mu} = \mu P \gamma \left(\frac{1}{\varepsilon-1}\right) + \mu g \\
\Rightarrow \quad & -\frac{\dot{\mu}}{\mu} = P\frac{\gamma}{\varepsilon-1} + g - \rho
\end{aligned}
$$

Solution

If we growthify the first order condition, we find (assuming log utility)

$$\begin{aligned}
& -g_c + \frac{g}{\varepsilon - 1} = \frac{\dot{\mu}}{\mu} + g \\
\Rightarrow \quad & -\frac{\dot{\mu}}{\mu} = g_c - \frac{g}{\varepsilon - 1} + g = g_P + g
\end{aligned}$$

Combining this with the result from the first order condition yields

$$\rho + g_P = P \frac{\gamma}{\varepsilon - 1}$$

In steady state $g_P = 0$, so we can then solve to find

$$\hat{R} = 1 - (\varepsilon - 1) \frac{\rho}{\gamma}$$

Dynamics

In the general case, we can express the prior equation as a quadratic form

$$\begin{aligned} \frac{\dot{R}}{1-R} &= \rho - (1-R)\frac{\gamma}{\varepsilon - 1} \\ \Rightarrow \quad \dot{R} &= -\frac{\gamma}{\varepsilon - 1}(R-1)(R-\hat{R}) \end{aligned}$$

As in the equilibrium, this is unstable around $\hat{R}$ and the only way to avoid violating either feasibility or transversality is to choose $\hat{R}$ exactly, so there are no dynamics

Steady State

This implies that the steady state growth rate $g = \gamma R$ is

$$\hat{g} = \gamma - (\varepsilon - 1)\rho$$

Recall that the equilibrium growth rate was

$$g^* = \frac{\gamma - (\varepsilon - 1)\rho}{\varepsilon} = \frac{\hat{g}}{\varepsilon} < \hat{g}$$

So the growth rate is inefficiently low in equilibrium

Efficiency Analysis

Consider the constrained problem where the social planner is choosing R subject to $\ell_i = P/N$

$$\begin{aligned}\rho U &= \frac{g_c}{\rho} + \log(c) \\ &= \frac{\gamma}{\varepsilon - 1}\frac{R}{\rho} + \log(1 - R)\end{aligned}$$

The first order condition for this problem in R yields

$$SMB = \frac{\gamma}{\varepsilon - 1}\frac{1}{\rho} = \frac{1}{1 - \hat{R}} = SMC$$

Private Incentives

The equivalent comparison for the equilibrium can be expressed as

$$PMB = \frac{\gamma}{\varepsilon - 1}\frac{1}{\rho + g} = \frac{1}{1 - R}$$

The additional "g" term reflects the fact that when firms create new products, they reduce the profits of existing firms slightly and thus this blunts incentives for innovation relative to the social optimum

This can be corrected by introducing a research subsidy that equalizes these conditions when at the optimal allocations

$$1 - s = \frac{\rho}{\rho + \hat{g}} = \frac{\rho}{\gamma - (\varepsilon - 2)\rho}$$

Schumpeterian Growth

Creative Destruction

In Romer-style model, new products impinged on incumbent profits, since ratio of profit/value to wages (opportunity cost) fell like $1/N$

A Schumpeterian dynamic features much more head-to-head competition: new products will directly displace incumbent products

For now, we'll keep the set of products fixed at $N = 1$ and let firms compete over who produces them

- You could think of these as "needs" of consumers rather than products

Productivity Gains

Suppose that producers have some labor productivity q_i so that

$$y_i = q_i \ell_i$$

When an innovation occurs, rather than producing a new product, we instead improve the productivity of making an existing product so that

$$q_i \to \lambda q_i$$

Where $\lambda > 1$ is called the innovation "step size"

Logarithmic Limit

What happens in the current model when $\varepsilon \to 1$? Turns out

$$y = \exp\left(\int_0^N \log(y_i) di\right)$$

But, notice that when $N \neq 1$, this does not satisfy constant returns to scale!

$$y(x \cdot \vec{y}) = x^N y(\vec{y})$$

So when we do assume $N = 1$, we can also jump to $\varepsilon = 1$ to make things simpler

Limit Pricing

Why is $\varepsilon = 1$ simpler? In a Schumpeterian world, new innovators are competing against incumbent producers

Recall that when $\varepsilon = 1$, the demand function will satisfy $y = p_i y_i$. Suppose the new innovator sets $p_i = \lambda w / q_i$ and the incumbent is producing a small positive quantity

Then the old incumbent ($-i$), who has productivity $q_{-i} = q_i / \lambda$ will get

$$\pi_{-i} = p_i y_i - w\ell_i = \left(\frac{\lambda w}{q_i}\right)\left(\frac{q_i}{\lambda}\ell_i\right) - w\ell_i = 0$$

So they will choose to simply not enter the market

Profit Levels

In this case, the new innovator will reap profits of

$$\pi_i = p_i y_i - w\ell_i = y - \frac{w}{q_i}\frac{1}{p_i} = \left(1 - \frac{1}{\lambda}\right) y$$

So when their technological lead is nill ($\lambda = 1$) the get no profits, and as their lead grows ($\lambda \to \infty$) it approaches y

Note that in this case the implied physical outcomes are

$$y_i = \frac{q_i}{\lambda w} y \quad \text{and} \quad \ell_i = \frac{y}{\lambda w}$$

Income Accounting

Labor market clearing for production implies

$$P = \frac{y}{\lambda w} \quad \Rightarrow \quad \frac{wP}{y} = \frac{1}{\lambda}$$

Similarly, note that the profit equation immediately implies

$$\frac{\pi}{y} = 1 - \frac{1}{\lambda}$$

So then this is an income accounting equation and

$$y = wP + \pi = w + [\pi - wR]$$

Productivity Aggregate

Given $\ell_i = y/(\lambda w)$, we can use the final good aggregator

$$w = \frac{Q}{\lambda} \quad \text{where} \quad \log(Q) = \int_0^1 \log(q_i) di$$

Using $\ell_i = P$, we can also derive an expresion for total output

$$y = QP$$

Here Q is an index of overall productivity whose growth rate we will refer to simply by g

Value Function

With creative destruction, finding the flow value of innovation slightly trickier

— with Poisson flow rate τ a new innovator displaces the incumbent

In general, a Poisson distribution with mean μ has the form

$$f(x|\mu) = \frac{\mu^x e^{-\mu}}{x!}$$

A Poisson process with rate τ is one in which the number of occurrences of an event over a time period Δ is Poisson distributed with mean $\Delta\tau$

Properties of Poisson

The parameter $\mu > 0$ describes both the mean and variance (evenly dispersed)

$$\mathbb{E}[x] = \mathrm{Var}(x) = \mu$$

Closed under linear combinations: the sum of two Poisson distributed RVs is Poisson (where the $\mu = \mu_1 + \mu_2$)

We can think of the likelihood as a proportional error term

$$\frac{\partial \log(f(x|\mu))}{\partial \mu} \sim \frac{x - \mu}{\mu}$$

Compare this to a Normal RV which looks like $x - \mu$

Small Time Steps

In our case, we are interested in the probability that one or more innovations occur

$$P(>0|\Delta\tau) = 1 - P(0|\Delta\tau) = 1 - \exp(-\Delta\tau) \approx \Delta\tau$$

Thus the value function will satisfy the approximate equation

$$\begin{aligned} v(t) &= \Delta\pi(t) + \exp(-\Delta r)(1 - \Delta\tau)v(t + \Delta) \\ &\approx \Delta\pi(t) + (1 - \Delta(r + \tau))v(t + \Delta) \end{aligned}$$

Follow familiar steps, in the limit as $\Delta \to 0$ we find

$$(r + \tau)v - \dot{v} = \pi$$

Free Entry

Our research production function is assumed to be

$$\tau = \gamma R \quad \Rightarrow \quad \gamma v = w$$

Rearranging the value function and using $g_v = g_w = g$, we find

$$v = \frac{\pi}{r + \tau - g_v} = \frac{\pi}{\rho + \tau}$$

Combining what we know yields the expression

$$\begin{aligned} \frac{\gamma\pi}{\rho + \tau} = w \quad &\Rightarrow \quad \gamma(\lambda - 1)P = \rho + \gamma R \\ &\Rightarrow \quad R^* = \frac{(\lambda - 1) - \rho/\gamma}{\lambda} \end{aligned}$$

Growth Rates

Working backwards, we find innovation rates of

$$\tau = \frac{\gamma(\lambda - 1) - \rho}{\lambda}$$

For the growth rate, consider the effect on Q over a time step Δ

$$\begin{aligned}\log(Q(t+\Delta)) &= \int_0^1 \log(q_i(t+\Delta))di \\ &= \int_0^1 \left[(\Delta\tau)\log(\lambda q_i(t)) + (1-\Delta\tau)\log(q_i(t))\right]di \\ &= \log(Q(t)) + \Delta\tau\log(\lambda)\end{aligned}$$

Thus as $\Delta \to 0$ we find $g = \frac{\partial \log(Q)}{\partial t} = \log(\lambda)\tau$