Chapter 4
Computation

Econ 3070: Macroeconomics 2.0

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Baseline

For comparison, let's consider the neoclassical growth model social planner in discrete time

$$v(k) = \max_i \left\{u(f(k) - i) + \beta v((1-\delta)k + i)\right\}$$

This has the optimality conditions

$$\begin{aligned} u'(c) &= \beta v_k(k') \\ v_k(k) &= u'(c)f'(k) + \beta(1-\delta)v_k(k') \end{aligned}$$

Combining these yields

$$\frac{u'(c)}{u'(c')} = \beta\left[f'(k) + (1-\delta)\right]$$

Solution

In steady state, we have

$$f'(k^*) = \frac{1-\beta}{\beta} + \delta$$
$$c^* = f(k^*) - \delta k^*$$

We can solve the dynamics with value function iteration

$$v^{r+1}(k) = \max_i \left\{ u(f(k) - i) + \beta v^r((1-\delta)k + i) \right\}$$

Then hope for convergence

$$\lim_{r \to \infty} v_r = v^*$$

Continuous Limit

Now let's take this to the limit where timesteps Δ are small

$$\begin{aligned} v(k,t) &= \max_i \{\Delta u(f(k)-i) + \exp(-\rho\Delta) v((1-\Delta\delta)k + \Delta i, t+\Delta)\} \\ &\approx \max_i \Big\{ \Delta u(f(k)-i) + v(k,t) \\ &\qquad + \Delta\big[-\rho v(k,t) + (i-\delta k) v_k(k,t) + \dot{v}(k,t) \big] \\ &\quad \Big\} \end{aligned}$$

Cancelling $v(k,t)$ and Δ on both sides yields the limit

$$\rho v(k,t) - \dot{v}(k,t) = u(f(k) - i(k,t)) + (i(k,t) - \delta k) v_k(k,t)$$

where $i(k,t)$ represents the optimal investment choice

Euler Equation

Taking the derivative with respect to i, the optimality condition is

$$\Delta u'(c) = \Delta \exp(-\rho\Delta) v_k((1-\Delta\delta)k + \Delta i, t+\Delta)$$
$$\lim_{\Delta \to 0} \to u'(c) = v_k(k,t)$$

Similarly, the envelope condition is

$$v_k(k) = \Delta u'(c) f'(k) + \exp(-\rho\Delta)(1-\Delta\delta) v_k(1-\Delta\delta)k + \Delta i, t+\Delta)$$

Using a similar linear expansion in Δ, we find in the limit of $\Delta \to 0$

$$(\rho+\delta) v_k(k,t) = u'(c) f'(k) + (i - \delta k) v_{kk}(k,t) + \dot{v}_k(k,t)$$

Dynamic Equations

Note that the envelope condition can be expressed as

$$
\begin{aligned}
(\rho+\delta)v_k(k) &= u'(c)f'(k) + \frac{d}{dt}\left[v_k(k,t)\right] \\
\Rightarrow (\rho+\delta)u'(c) &= f'(k)u'(c) + \frac{d}{dt}\left[u'(c)\right] \\
\Rightarrow (\rho+\delta)u'(c) &= f'(k)u'(c) + \dot{c}\,u''(c)
\end{aligned}
$$

Rearranging this, we find the familiar system

$$
\begin{aligned}
\dot{k} &= f(k) - \delta k - c \\
\frac{\dot{c}}{c} &= \frac{1}{\varepsilon_u(c)}\left[f'(k) - (\rho+\delta)\right]
\end{aligned}
$$