Chapter 4

Value Functions

Econ 3070: Macroeconomics 2.0

University of Pittsburgh, 2023

Baseline

For comparison, let's consider the neoclassical growth model social planner in discrete time

$$v(k) = \max_i \{u(f(k) - i) + \beta v((1-\delta)k + i)\}$$

This has the optimality conditions

$$u'(c) = \beta v_k(k')$$
$$v_k(k) = u'(c)f'(k) + \beta(1-\delta)v_k(k')$$

Combining these yields

$$\frac{u'(c)}{u'(c')} = \beta\left[f'(k) + (1-\delta)\right]$$

Solution

In steady state, we have

$$f'(k^*) = \frac{1-\beta}{\beta} + \delta$$
$$c^* = f(k^*) - \delta k^*$$

We can solve the dynamics with value function iteration

$$v^{r+1}(k) = \max_i \{u(f(k) - i) + \beta v^r((1-\delta)k + i)\}$$

Then hope for convergence

$$\lim_{r\to\infty} v_r = v^*$$

Continuous Limit

Now let's take this to the limit where timesteps Δ are small

$$
\begin{aligned}
v(k,t) &= \max_i \{\Delta u(f(k)-i) + \exp(-\rho\Delta) v((1-\Delta\delta)k + \Delta i, t+\Delta)\} \\
&\approx \max_i \Big\{\Delta u(f(k)-i) + v(k,t) \\
&\qquad + \Delta\Big[-\rho v(k,t) + (i-\delta k) v_k(k,t) + \dot{v}(k,t)\Big] \\
&\quad\Big\}
\end{aligned}
$$

Cancelling $v(k,t)$ and Δ on both sides yields the limit

$$
\rho v(k,t) - \dot{v}(k,t) = u(f(k) - i(k,t)) + (i(k,t) - \delta k) v_k(k,t)
$$

where $i(k,t)$ represents the optimal investment choice

Euler Equation

Taking the derivative with respect to i, the optimality condition is

$$\Delta u'(c) = \Delta \exp(-\rho\Delta) v_k((1-\Delta\delta)k + \Delta i, t+\Delta)$$

$$\lim_{\Delta\to 0} \to u'(c) = v_k(k,t)$$

Similarly, the envelope condition is

$$v_k(k) = \Delta u'(c) f'(k) + \exp(-\rho\Delta)(1-\Delta\delta) v_k(1-\Delta\delta)k + \Delta i, t+\Delta)$$

Using a similar linear expansion in Δ, we find in the limit of $\Delta \to 0$

$$(\rho+\delta) v_k(k,t) = u'(c) f'(k) + (i-\delta k) v_{kk}(k,t) + \dot{v}_k(k,t)$$

Dynamic Equations

Note that the envelope condition can be expressed as

$$
\begin{aligned}
(\rho+\delta)v_k(k) &= u'(c)f'(k) + \frac{d}{dt}\left[v_k(k,t)\right] \\
\Rightarrow (\rho+\delta)u'(c) &= f'(k)u'(c) + \frac{d}{dt}\left[u'(c)\right] \\
\Rightarrow (\rho+\delta)u'(c) &= f'(k)u'(c) + \dot{c}\, u''(c)
\end{aligned}
$$

Rearranging this, we find the familiar system

$$
\begin{aligned}
\dot{k} &= f(k) - \delta k - c \\
\frac{\dot{c}}{c} &= \frac{1}{\varepsilon_u(c)}\left[f'(k) - (\rho+\delta)\right]
\end{aligned}
$$

Stochastic Optimization

Most stochastic elements in discrete time models have straightforward analogues in continuous time

There are number of useful processes that are useful in a continuous time context because they preserve locality

The cost of introducing these is often the inclusion of higher order derivatives in value function equations

Wiener Process

Bedrock process is the Wiener process / Brownian motion, characterized by $W_0 = 0$ and

1. Independent, normal increments

$$W_s - W_t \mid W_{[0,t]} \sim \mathcal{N}(0, s-t)$$

2. Continuous with probability one

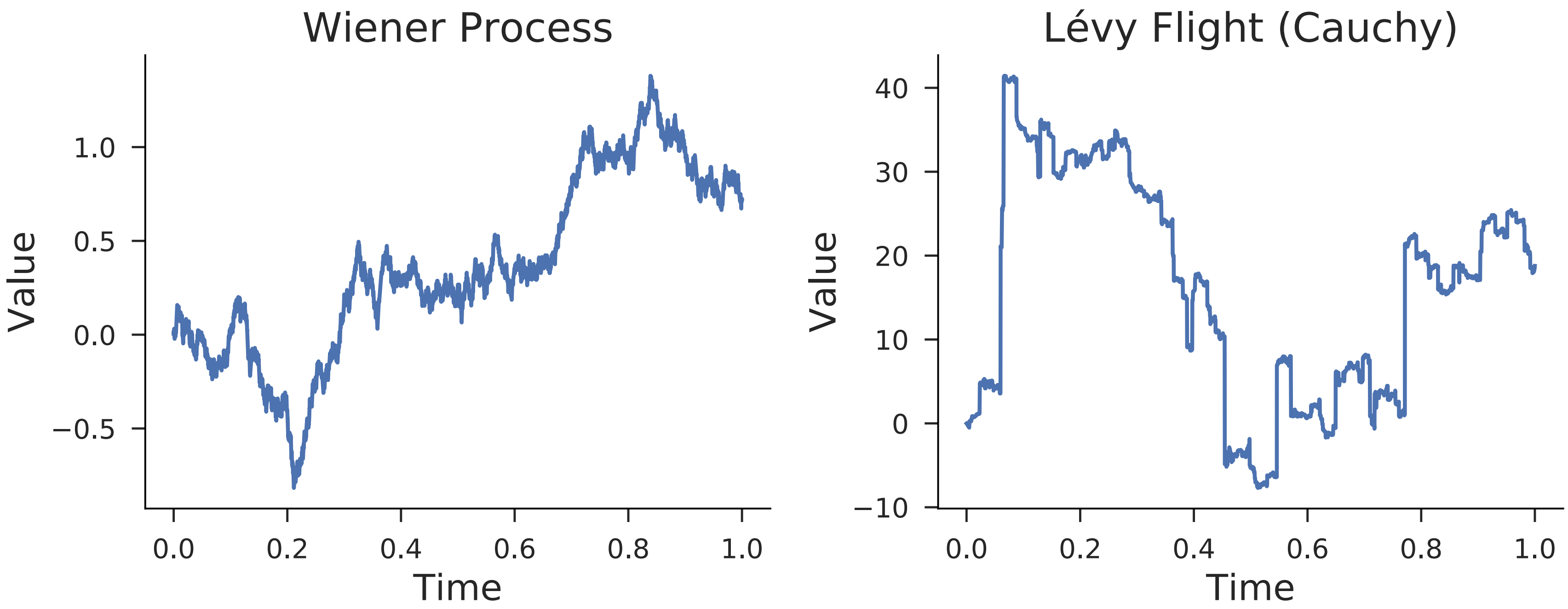
$$\lim_{h \to 0} \mathbb{P}(|W_{t+h} - W_h| > \varepsilon) = 0$$

Surprisingly few other options with just (1) as increments must be mean zero and Levy stable

— Cauchy would be another example, but this has undefined mean and variance

Continuous Processes

Wiener process is the continuous limit of a random walk with increments $\mathcal{N}(0, \Delta)$ as $\Delta \to 0$



Central Limit Theorem

It turns out that *any* standard random walk with finite mean and variance increments converges to a Wiener process

$$\frac{1}{\sqrt{N}} \sum_{i=0}^{N} z_i \xrightarrow[N]{} \text{Wiener process}$$

Note that Cauchy distribution has infinite variance so Levy Flight does not obey Central Limit Theorem

Random walks are also *recurrent*: they return to the same neighborhood infinitely often in the future. This breaks down for $d \geq 3$ though.

Characteristic Function

The characteristic function of a distribution is a spectral decomposition closely related to the Fourier transform

$$\varphi_X(t) = \mathbb{E}[\exp(itX)] = \int_{-\infty}^{\infty} \exp(itx) f(x) dx$$

This has the distinctive property that it generates the moments of the distribution

$$\mathbb{E}[X^n] = i^{-n} \left[\frac{d^n}{dt^n} \varphi_X(t) \right]_{t=0}$$

Spectral Convolution

We can also express the convultion of two variables in characteristic space. Suppose that X and Y are independent and $Z = X + Y$ is distributed as

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx$$

In terms of characteristic functions, convolution simply becomes multiplication as in

$$\varphi_Z(t) = \varphi_X(t) \cdot \varphi_Y(t)$$

Gaussian Basin

Taking a Taylor expansion, we can see that in general

$$\varphi_X(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mathbb{E}[X^n]$$

Suppose that $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] = 1$. Consider the distribution of $Z = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$

$$\varphi_Z(t) = \left[\varphi_X\left(\frac{t}{\sqrt{n}}\right)\right]^n \approx \left(1 - \frac{\frac{1}{2}t^2}{n}\right)^n \longrightarrow \exp\left(-\frac{1}{2}t^2\right)$$

This coincides with the characteristic function of a standard normal, and so $Z \sim \mathcal{N}(0,1)$

Drift Diffusion Processes

We can embed this into a more general class known as drift diffusion processes

$$dX_t = \mu \cdot dt + \sigma \cdot dW_t$$

And even generate a continuous time analog of an AR(1) process (the Ornstein–Uhlenbeck process)

$$dX_t = \kappa(\mu - X_t) \cdot dt + \sigma \cdot dW_t$$

Discrete Events

More general drift diffusion process is represented by class of jump diffusion processes.

We can ditch the continuity assumption but preserve independent increments by bolting on a Poisson jump process.

$$dX_t = \sigma \cdot dW_t + dJ_t$$

Parameter λ determines jump rate. Size of jumps can even be random as long as i.i.d.

Itô's Lemma

Consider the drift process $dx = \mu dt + \sigma dW$. For small Δ, this evolves according to

$$x(t + \Delta) = x(t) + \Delta\mu + \sqrt{\Delta}\sigma z$$

where $z \sim \mathcal{N}(0, 1)$

We can then approximate v by taking a Taylor expansion

$$\begin{aligned}
& v(x(t + \Delta), t + \Delta) - v(x(t), t) \\
& \quad \approx \Delta \dot{v}(x(t), t) + \mathbb{E}[(\Delta\mu + \sqrt{\Delta}\sigma z)] v_x(x(t), t) \\
& \qquad + \frac{1}{2}\mathbb{E}[(\Delta\mu + \sqrt{\Delta}\sigma z)^2] v_{xx}(x(t), t) \\
& \quad \approx \Delta \left[\dot{v}(x(t), t) + \mu v_x(x(t), t) + \frac{\sigma^2}{2} v_{xx}(x(t), t) \right]
\end{aligned}$$

Value Function

The general expression for a recursive form value function is

$$
\begin{aligned}
&v(x(t), t) \approx \Delta u(x(t)) + \exp(-\rho\Delta)\mathbb{E}\left[v(x(t+\Delta), t+\Delta)\right] \\
\Rightarrow\ &v(x, t) \approx \Delta u(x) + v(x, t) \\
&\quad + \Delta\left[-\rho v(x, t) + \dot{v}(x, t) + \mu v_x(x, t) + \frac{\sigma^2}{2} v_{xx}(x, t)\right]
\end{aligned}
$$

Eliminating Δ and dropping explicit dependence yields the value equation

$$
\rho v - \dot{v} = u(x) + \mu v_x + \frac{\sigma^2}{2} v_{xx}
$$

Distributions

We can apply similar techniques to distribution functions. We will denote the cumulative distribution of a variable x by $F(x)$

Consider the case of a zero-centered O-U process with

$$dx = -\kappa(x - \mu)dt + \sigma dW$$

We can derive conditions for the evolution of the cumulative density F over x: these results are sometimes called the **Fokker-Planck** or **Kolmogorov Forward Equations**

Discrete Limit

Let's approach this again using discrete approximations (note inverted signs)

$$
\begin{aligned}
F(x,t) &\approx \mathbb{E}\left[F(x + \Delta\kappa(x-\mu) - \sqrt{\Delta}\sigma z, t-\Delta)\right] \\
&\approx F(x,t) - \dot{F}(x,t)\Delta + \mathbb{E}[\Delta\kappa(x-\mu) - \sqrt{\Delta}\sigma z]F_x(x,t) \\
&\quad + \frac{1}{2}\mathbb{E}[(\Delta\kappa(x-\mu) - \sqrt{\Delta}\sigma z)^2]F_{xx}(x,t) \\
&\approx F(x,t) + \Delta\left[-\dot{F}(x,t) + \kappa(x-\mu)\sigma F_x(x,t) + \frac{\sigma^2}{2}F_{xx}(x,t)\right]
\end{aligned}
$$

Thus we arrive at the desired result

$$
\dot{F} = \kappa(x-\mu)F_x + \frac{\sigma^2}{2}F_{xx}
$$

O-U Limiting Distribution

In the limit, we will have $\dot{F} = 0$ so that (here $f = F_x$)

$$\kappa(x - \mu)F_x = -\frac{\sigma^2}{2}F_{xx} \quad \Rightarrow \quad \kappa(x - \mu)f = -\frac{\sigma^2}{2}f_x$$

We can solve the prior equation using the guess $\mathcal{N}(\mu, \gamma)$

$$f(x) = \frac{1}{\sqrt{2\pi\gamma^2}} \exp\left(-\frac{1}{2}\left(\frac{x - \mu}{\gamma}\right)^2\right)$$

Canceling common terms, this yields the following

$$\kappa = \frac{\sigma^2}{2\gamma^2} \quad \Rightarrow \quad \gamma^2 = \frac{\sigma^2}{2\kappa}$$

Extending NCG

Consider the case of stochastic productivity z as an O-U process

$$f(x,k) = e^x k^\alpha$$

The derived value function is then (same FOC as before)

$$\rho v = u(e^x k^\alpha - i) + (i - \delta k)v_k - \kappa(x-\mu)v_x + \frac{\sigma^2}{2}v_{xx}$$

We can characterize the stationary distribution over (x,k) with

$$(i - \delta k)F_k = \kappa(x-\mu)F_x + \frac{\sigma^2}{2}F_{xx}$$

Discrete Events

Stochastic processes can also have discrete jumps where the value changes discontinuously

Easiest formulation is to use a Poisson process to model event occurrence. For Poisson *distribution*, probability of k occurrences for parameter λ

$$P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

For a Poisson *process*, number of occurrences over time step Δ is Poisson distributed with parameter $\lambda = \Delta p$, where $p > 0$ is a **flow probability**

Working With Poisson

Probability of k occurrences with flow probability p over step Δ

$$P(k|p,\Delta) = \frac{(\Delta p)^k e^{-\Delta p}}{k!}$$

For small Δ only the $k = 0$ and $k = 1$ values matter

$$\begin{aligned}
P(0|p,\Delta) &= \exp(-\Delta p) \approx 1 - \Delta p \\
P(1|p,\Delta) &= \Delta p \exp(-\Delta p) \approx \Delta p \\
P(2|p,\Delta) &= \frac{1}{2}(\Delta p)^2 \exp(-\Delta p) \approx \frac{1}{2}\Delta^2 p \\
P(k|p,\Delta) &= \frac{1}{k!}(\Delta p)^k \exp(-\Delta p) \approx \frac{1}{k!}\Delta^k p
\end{aligned}$$

Jump Processes

Using a Poisson process means that the time between occurrences is exponentially distributed (with mean $1/p$)

$$P(k > 0|p, \Delta) = 1 - P(0|p, \Delta) = 1 - \exp(-\Delta p)$$

Because the conditional expectation of an exponential is itself exponential, there is no sense in which we can be "due" for a jump, they are really independent over time

When the jump occurs, the value of the stochastic process increments by some (possibly random) value. Simplest case is we just increment a counter that tracks the number of occurrences

Using in Value Function

Suppose we want to find the present value of the "counter" jump process with flow probability p, meaning for state n we have $u(n) = n$

$$
\begin{aligned}
v(n,t) &= \Delta n + (1-\Delta\rho)\left[\Delta p v(n+1,t+\Delta) + (1-\Delta p) v(n,t+\Delta)\right] \\
v(n,t) - v(n,t+\Delta) &= \Delta\left[n + p(v(n+1,t+\Delta) - v(n,t+\Delta))\right] \\
\rho v(n,t) - \dot{v}(n,t) &= n + p\left[v(n+1,t) - v(n,t)\right]
\end{aligned}
$$

So we end up with a term that is the flow probability time the value increment resulting from an occurrence.

This generalizes to almost any situation, and two Poisson processes will "never" hit at the same time, so we don't need to worry about interaction terms

Solving Value Function

There's still more we can do from the previous equation. Consider the candidate solution $v(n, t) = A + Bn$

$$\rho(A + Bn) = n + pB$$

Matching coefficients, we can see that

$$\rho B = 1 \quad \Rightarrow \quad B = \frac{1}{\rho}$$
$$\rho A = pB \quad \Rightarrow \quad A = \frac{p}{\rho}B = \frac{p}{\rho^2}$$

Combining these then yields the solution

$$v(n) = \frac{n}{\rho} + \frac{p}{\rho^2}$$