Chapter 3

Ramsey Model

Econ 3070: Macroeconomics 2.0

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Motivation

Solow model assumes constant exogenous saving rate

Questions:

- what determines the saving rate?
- when agents decide saving in a competitive equilibrium environment, do they choose a Pareto optimal saving rate?

Objective of this chapter: construct a general equilibrium model to analyze aggregate saving

Assumptions

Modeling choices:

- number of agent $\rightarrow$ single representative
- time horizon $\rightarrow$ finite / infinite
- decision maker $\rightarrow$ planner / decentralized

Simplest consumption-saving model:

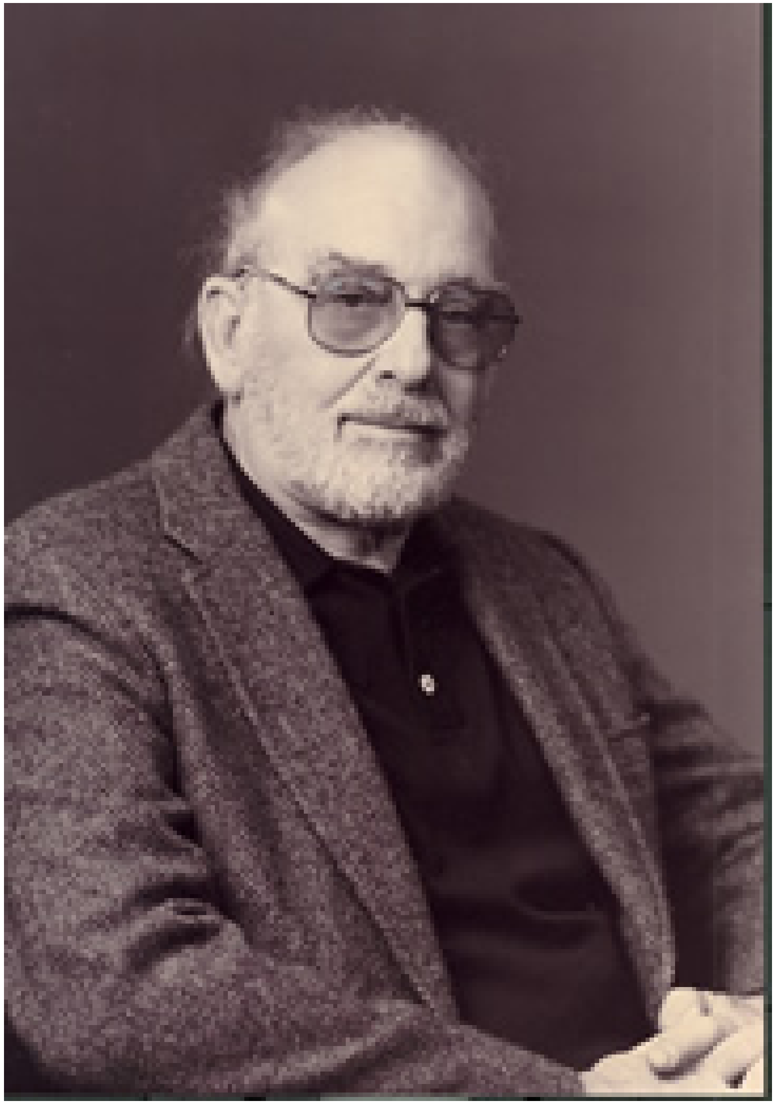
- Single agent, finite horizon, no depreciation

Ramsey-Cass-Koopmans Model

Frank Plumpton Ramsey (1903-1930)

Tjalling Koopmans (1910-1986)

David Cass (1937-2008)



General of the Army

Agents

Population grows at rate n and $L(0) = 1$ so that $L(t) = \exp(nt)$

Objective function of each agent at $t = 0$

$$\begin{aligned} U(0) &= \int_0^\infty \exp(-\rho t) u(c(t)) L(t) dt \\ &= \int_0^\infty \exp(-(\rho - n)t) u(c(t)) dt \end{aligned}$$

Utility from c_t in t evaluated at time 0 is $\exp(-\rho t) u(c_t)$

Assumption $\to \rho > n$

Discounting

Where does the exponential form of discounting come from?

— intuition: calculate the value of $1 in t periods

Divide interval $[0, t]$ into t/Δ equally-sized subintervals

Discount rate in each subinterval is $\Delta\rho \cdot r$

Value of 1 util, t periods from now

$$v(t \mid \Delta) = (1 - \Delta \cdot \rho)^{t/\Delta}$$

Continuous limit

Letting $\Delta \to 0$

$$v(t) = \lim_{\Delta \to 0} v(t|\Delta) \equiv \lim_{\Delta \to 0} (1 - \Delta \cdot \rho)^{t/\Delta}$$

Use L'Hôpital's rule:

$$\begin{aligned} v(t) &= \exp\left[\lim_{\Delta \to 0} \log(1 - \Delta \cdot \rho)^{t/\Delta}\right] \\ &= \exp\left[\lim_{\Delta \to 0} \frac{\log(1 - \Delta \cdot \rho)}{\Delta/t}\right] \\ &= \exp\left[\lim_{\Delta \to 0} \frac{-\rho/(1 - \Delta\rho)}{1/t}\right] \\ &= \exp(-\rho t) \end{aligned}$$

Budget constraint

Agents earn a wage $w(t)$ at every instant and can borrow and save assets $a(t)$, yielding interesting rate $r(t)$

Per-period budget constraint

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - c(t)L(t)$$

and in per capita terms:

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t)$$

Time varying coefficients

Math refresh 3: Linear *non-autonomous* differential equations

$$\dot{x}(t) = m(t)x(t) + b(t)$$

Steady state (may not exist): $x(t) = -b(t)/m(t)$

General solution

$$x(t) = \left[d + \int_0^t b(s) \exp\left(-\int_0^s m(v)dv\right) ds\right] \exp\left(\int_0^t m(s)ds\right)$$

Boundary condition $d = x(0) = x_0$ yields solution

$$x(t) = \left[x_0 + \int_0^t b(s) \exp\left(-\int_0^s m(v)dv\right) ds\right] \exp\left(\int_0^t m(s)ds\right)$$

Lifetime budget constraint

In our case

- $m(t) = r(t) - n$
- $b(t) = w(t) - c(t)$

This yields the solution

$$a(t) = \left[a_0 + \int_0^t [w(s) - c(s)] \exp\left(-\int_0^s (r(v) - n) dv\right) ds\right]$$
$$\times \exp\left(\int_0^t (r(s) - n) ds\right)$$

Limiting conditions

No-Ponzi condition:

$$\lim_{t \to \infty} a(t) \exp\left(-\int_0^t (r(s) - n) ds\right) = 0$$

Take limit when $T \to \infty$ of T-period lifetime budget constraint to obtain

$$a_0 = \int_0^\infty [c(t) - w(t)] \exp\left(-\int_0^t (r(s) - n) ds\right) dt$$

Optimization problem

How to solve the optimization problem?

- use per-period budget constraint
- construct Hamiltonian, the continuous-time analog to Lagrangian
- apply maximum principle from optimal control theory

Agent's problem

$$\max_{[a(t),c(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp(-(\rho-n)t)u(c(t))dt$$

$$\text{subj. to} \quad \dot{a}(t) = (r(t)-n)a(t) + w(t) - c(t)$$

Optimal control

Consumer lives between periods 0 and 1

$$\begin{aligned} \max_{[a(t),c(t)]_{t=0}^{1}} & \int_0^1 \exp(-\rho t)u(c(t))dt \\ \text{subj. to} \quad & \dot{a}(t) = ra(t) + w - c(t) \\ & a(t) \geq 0 \quad \text{and} \quad a(0) = a_0 > 0 \end{aligned}$$

Lagrangian

$$\begin{aligned} \mathcal{L} = & \int_0^1 \exp(-\rho t)u(c(t))dt \\ & + \int_0^1 \lambda(t)[ra(t) + w - c(t) - \dot{a}(t)]dt \end{aligned}$$

Finite solution

Integration by parts

$$\int_0^1 \lambda(t)\dot{a}(t)dt = \lambda(1)a(1) - \lambda(0)a(0) - \int_0^1 \dot{\lambda}(t)a(t)dt$$

Optimality conditions

$$\exp(-\rho t)u'(c(t)) = \lambda(t)$$

$$\dot{\lambda}(t) = -r\lambda(t)$$

Alternatively with $\mu(t) = \exp(\rho t)\lambda(t)$

$$u'(c(t)) = \mu(t)$$

$$\rho\mu(t) - \dot{\mu}(t) = r\mu(t)$$

Infinite horizon

More general infinite-horizon optimal control problem:

$$\max_{\vec{x},\vec{y}} W(\vec{x},\vec{y}) \equiv \int_0^\infty \exp(-\rho t) f(t, x(t), y(t)) dt$$

subject to

$$\dot{x}(t) = g(t, x(t), y(t))$$

and

$$x(0) = x_0 \quad \text{and} \quad \lim_{t\to\infty} x(t) \geq x_1$$

Notice payoff f depends on time only through exponential discounting

Hamiltonian function

Hamiltonian

$$H(t,x,y,\lambda) = \exp(-\rho t) f(t,x,y) + \lambda g(t,x,y)$$

Current-value Hamiltonian

$$\hat{H}(t,x,y,\mu) = f(t,x,y) + \mu g(t,x,y)$$

In general, we will work only with the current-value Hamiltonian for the sake of simplicity, and we will often omit the "hat"

Maximum Principle

Theorem: *Maximum Principle, discounted infinite-horizon problems*

Let $\hat{H}(t, x, y, \mu)$ be the current-value Hamiltonian of the typical discounted infinite-horizon optimal control problem. Then the optimal control pair $(\hat{x}(t), \hat{y}(t))$ satisfies the following necessary conditions

$$\hat{H}_y\left(t, \hat{x}(t), \hat{y}(t), \mu(t)\right) = 0 \; \forall\, t \in \mathbb{R}_+$$

$$\rho\mu(t) - \dot{\mu}(t) = \hat{H}_x\left(t, \hat{x}(t), \hat{y}(t), \mu(t)\right) \; \forall\, t \in \mathbb{R}_+$$

$$\dot{x}(t) = \hat{H}_\mu\left(t, \hat{x}(t), \hat{y}(t), \mu(t)\right) \; \forall\, t \in \mathbb{R}_+$$

$$x(0) = x_0 \quad \text{and} \quad \lim_{t\to\infty} x(t) \geq x_1$$

and the transversality condition holds (see next slide)

Transversality Condition

The transversality condition in the general case is

$$\lim_{t\to\infty} \exp(-\rho t)\hat{H}(t, \hat{x}(t), \hat{y}(t), \mu(t)) = 0$$

However, when f and g are assumed to be weakly monotone in x and y, and f is bounded in t, the we can simplify this to

$$\lim_{t\to\infty} \exp(-\rho t)\mu(t)\hat{x}(t) = 0$$

In the economic setting later, this will be akin to assuming that the present value of accumulated assets must be zero in the limit

Initution

The "co-state" variable μ tracks the value of an additional increment of the state variable x

$$\mu(t) = \int_0^\infty \exp(-\rho s) \hat{H}_x(t+s) ds$$

Above satisfies co-state evolution equation (integration by parts)

$$\begin{aligned}
\dot{\mu}(t) &= \int_0^\infty \exp(-\rho s) \dot{\hat{H}}_x(t+s) dx \\
&= \Big[\exp(-\rho s) \hat{H}_x(t+s)\Big]_0^\infty - \rho \int_0^\infty \exp(-\rho s) \hat{H}_x(t+s) ds \\
&= -\hat{H}_x(t) + \rho \mu(t)
\end{aligned}$$

Notice we need an envelope condition for this interpretation to make sense

Application to Ramsey

Current-value Hamiltonian for Ramsey model:

$$\hat{H}(a, c, \mu) = u(c) + \mu\left[(r-n)a + w - c\right]$$

Candidate solution:

$$0 = \hat{H}_c(a, c, \mu) = u'(c) - \mu$$

$$(\rho - n)\mu - \dot{\mu} = \hat{H}_a(a, c, \mu) = \mu(r - n)$$

$$\dot{a} = (r-n)a + w - c$$

$$\lim_{t \to \infty} \exp(-(\rho - n)t)\mu(t)a(t) = 0$$

$\hat{H}(a, c, \mu)$ is concave in (a, c) since it is the sum of a concave function of c and a linear function of (a, c)

Characterize solution

Second optimality condition

$$\frac{\dot{\mu}}{\mu} = -(r - \rho)$$

First optimality condition

$$u'(c) = \mu$$

taking the time derivative and dividing

$$\frac{u''(c)c}{u'(c)}\frac{\dot{c}}{c} = \frac{\dot{\mu}}{\mu} = -(r - \rho)$$

Thus we have eliminated μ and have a differential equation in c

Utility elasticity

Euler equation:

$$\frac{\dot{c}}{c} = \frac{1}{\varepsilon_u(c)}(r - \rho)$$

where

$$\varepsilon_u(c) \equiv -\frac{u''(c)c}{u'(c)}$$

Special case (CRRA):

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta} \quad \Rightarrow \quad \varepsilon_u(c) = \theta \quad \forall\, c$$

Consumption path

Implied consumption function:

$$c(t) = c(0) \exp\left(\int_0^t \frac{r(s) - \rho}{\varepsilon_u(c(s))} ds\right)$$

Let $\bar{r}(t)$ be the average interest rate between dates 0 and t

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(s) ds$$

Once again for the case of CRRA

$$c(t) = c(0) \exp\left(\left(\frac{\bar{r}(t) - \rho}{\theta}\right) t\right)$$

Initial condition

Where is $c(0)$ coming from? Recall the lifetime budget constraint

$$a_0 = \int_0^\infty [c(t) - w(t)] \exp(-(\bar{r}(t) - n)t)dt$$

Plugging in $c(t)$ for the case of CRRA utility

$$c(0) = \frac{a_0 + \int_0^\infty w(t) \exp\left(-\left(\bar{r}(t) - n\right)t\right)\ dt}{\int_0^\infty \exp\left(-\left(\frac{(\theta-1)\bar{r}(t)}{\theta} + \frac{\rho}{\theta} - n\right)t\right)\ dt}$$

Limiting conditions

Transversality condition ensures no-Ponzi schemes

Solution to μ equation

$$\begin{aligned}\mu(t) &= \mu(0)\exp\left(-\int_0^t (r(s)-\rho)ds\right)\\ &= u'(c(0))\exp\left(-\int_0^t (r(s)-\rho)ds\right)\end{aligned}$$

substituting into the transversality condition

$$\lim_{t\to\infty} a(t)\exp\left(-\int_0^t (r(s)-n)ds\right) = 0$$

Equilibrium

Optimization yields pair $(a(t), c(t))$ which implies a competitive equilibrium pair $(k(t), c(t))$

Since $a(t) = k(t)$, transversality condition is also equivalent to

$$\lim_{t\to\infty} k(t) \exp\left(-\int_0^t (r(s) - n)ds\right) = 0$$

Firms are standard:

$$r(t) = R(t) - \delta = f'(k(t)) - \delta$$

$$w(t) = f(k(t)) - f'(k(t))k(t)$$

Steady state

Steady-state equilibrium is an equilibrium path in which capital-labor ratio and consumption are constant, thus:

$$\dot{c} = 0 \quad \text{and} \quad \dot{k} = 0$$

Steady state Ramsey continuous time:

$$f'(k^*) = \rho + \delta$$

$$c^* = f(k^*) - (n + \delta)k^*$$

Compare with Solow model:

— population growth has no impact on k^*
— when $g = 0$, k^*, and c^* do not depend on u
— u does affect the transitional dynamics

Cobb-Douglas

Here we have $f(k) = k^\alpha$. Therefore

$$
\begin{aligned}
k^* &= \left(\frac{\alpha}{\rho + \delta}\right)^{\frac{1}{1-\alpha}} \\
c^* &= (k^*)^\alpha - (n + \delta)k^* \\
&= (k^*)^\alpha \left[1 - \alpha\left(\frac{n + \delta}{\rho + \delta}\right)\right]
\end{aligned}
$$

So we can see that the investment rate is

$$
s = \alpha\left(\frac{n + \delta}{\rho + \delta}\right) < \alpha
$$

Dynamics

Two differential equations:

$$\dot{k} = f(k) - (n+\delta)k - c$$
$$\frac{\dot{c}}{c} = \frac{1}{\varepsilon_u(c)}\left[f'(k) - \delta - \rho\right]$$

Initial condition on capital $k(0) = k_0 > 0$ and feasibility $k(t) \geq 0$

Boundary condition at infinity (transversality)

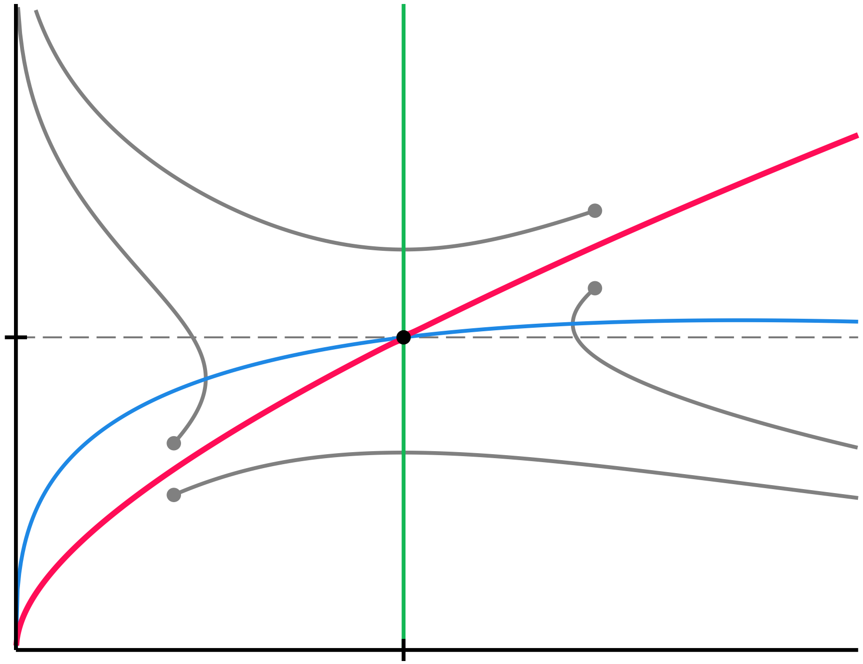
$$\lim_{t\to\infty} k(t)\exp\left(-\int_0^t \left(f'(k(s)) - \delta - n\right) ds\right) = 0$$

Goal is to set $c(0)$ so that boundary at infinity holds — this is surprisingly tough!

Phase diagram

$$\dot{c} = 0 \Rightarrow \qquad f'(k) = \delta + \rho$$

$$\dot{k} = 0 \Rightarrow \quad c = f(k) - (n + \delta)k$$



Uniqueness

Why is the stable arm unique?

- if $c(0)$ started below stable arm, capital would eventually reach the maximum level (zero consumption) $\bar{k} > k_{gold} \rightarrow$ violates transversality condition
- if $c(0)$ started above stable arm, eventually we would get $k < 0 \rightarrow$ violates feasibility

Stability

Theorem: *Saddle path stability for linear problems*

Consider the linear differential equation over $x \in \mathbb{R}^n$

$$\dot{x}(t) = Ax(t) + b$$

with initial value $x(0) = x_0$. Here A is an $n \times n$ matrix and b is a length n vector. Let x^* be the steady state, so that $Ax^* + b = 0$.

If $m \leq n$ of the eigenvalues of A have negative real parts, then there exists an m-dimensional subspace $M \subset \mathbb{R}^n$ such that for any $x(0) \in M$, there is a unique solution with $x(t) \to x^*$.

Eigenvalues

The steady state will satisfy $x^* = A^{-1}b$, and so we can define $\tilde{x} = x - x^*$ which will evolve according to

$$\dot{\tilde{x}} = A\tilde{x}$$

Since A is square, we can eigendecompose it with the matrix of eigenvectors V and the diagonal matrix of eigenvalues Λ, so that

$$A = V\Lambda V^{-1}$$

Then we can define $\hat{x} = V^{-1}\tilde{x}$ which evolves according to

$$\dot{\hat{x}} = \Lambda\hat{x}$$

But then this is a diagonal system! So we have $\dot{\hat{x}}_i = \lambda_i\hat{x}_i$ for all i

Stable manifold

Convergence of $\hat{x}_i$ depends on the sign of λ_i: positive means unstable, negative means stable

Thus any stable solution must have $\hat{x}_i(0) = 0$ for any i with $\lambda_i > 0$

So the stable manifold has dimensionality equal to number of negative λ_i

$$D = \#\left\{i \,\middle|\, \lambda_i < 0\right\}$$

In "hat" space this is simply a flat subspace, but it is rotated and shifted moving to "real" space

Non-linear systems

For the non-linear system

$$\dot{x}(t) = G(x(t))$$

we can define the Jacobian $J(x)$ of G and consider the matrix $A = J(x^*)$.

An analogous result follows for the linearized system

$$\dot{x}(t) = J(x^*)(x(t) - x^*)$$

Linearization

Linearizing system of differential equations around (k^*, c^*)

$$\frac{d(k-k^*)}{dt} \simeq (f'(k^*) - n - \delta)(k-k^*) - (c-c^*)$$

$$\frac{d(c-c^*)}{dt} \simeq \frac{c^* f''(k^*)}{\varepsilon_u(c^*)}(k-k^*) + \frac{f'(k^*) - \delta - \rho}{\varepsilon_u(c^*)}(c-c^*)$$

Jacobian is given by:

$$J(k^*, c^*) = \begin{bmatrix} \rho - n & -1 \\ \frac{c^* f''(k^*)}{\varepsilon_u(c^*)} & 0 \end{bmatrix}$$

Roots of associated characteristic polynomial confirm saddle-path

Technology

Let $\dot{A}/A = g$ and use production function

$$Y = F(K, AL)$$

As in Solow model define

$$y \equiv \frac{Y}{AL} \equiv f(k)$$

where

$$k \equiv \frac{K}{AL}$$

Existence

Is utility function consistent with existence of BGP?

Check Euler equation:

$$\frac{\dot{c}}{c} = \frac{1}{\varepsilon_u(c)}(r - \rho)$$

if $r(t) \to r^*$, then $\dot{c}(t)/c(t) \to g_c$ requires $\varepsilon_u(c(t)) \to \varepsilon_u$

Proposition: *Balanced growth in Ramsey model*

Balanced growth in the neoclassical model requires that asymptotically (as $t \to \infty$) all technological change is purely labor augmenting and the elasticity of intertemporal substitution, $\varepsilon_u(c(t))$, tends to a constant ε_u.

Example

Using CRRA preferences with $c(t) \equiv C(t)/L(t)$

$$\int_0^\infty \exp(-(\rho - n)t)\frac{c(t)^{1-\theta}}{1-\theta}dt$$

But as with $y(t)$, $c(t)$ also grows along the BGP. Bounded utility?

$$\tilde{c}(t) \equiv \frac{C(t)}{A(t)L(t)} \equiv \frac{c(t)}{A(t)}$$

Example

Lifetime utility

$$\begin{aligned} &\int_0^\infty \exp(-(\rho - n)t)\frac{c(t)^{1-\theta}}{1-\theta}dt \\ =&\int_0^\infty \exp(-(\rho - n - g(1-\theta))t)\frac{\tilde{c}(t)^{1-\theta}}{1-\theta}dt \end{aligned}$$

Assumption $\rightarrow \rho - n > g(1-\theta)$

Dynamics

Law of motion for capital in Ramsey model with technology growth:

$$\dot{k} = f(k) - \tilde{c} - (n + g + \delta)k$$

Euler equation:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\dot{c}}{c} - g = \frac{1}{\theta}(r - \rho - \theta g)$$

Steady state

Transversality condition:

$$\lim_{t\to\infty} k(t) \exp\left(-\int_0^t [f'(k(s)) - g - \delta - n] ds\right) = 0$$

Steady state:

$$f'(k^*) = \rho + \delta + \theta g$$

so now u plays a role in determining k^* and resulting output

Extensions

Role of policy

If ρ, δ, θ and g are same across countries, then the only theory of differences in $y^*(t)$ is differences in levels of $A(t)$

That is OK, but not very interesting because $A(t)$ is a residual

A possibility is to introduce differences in k^* across countries via differences in policy

Taxation

Introduce linear capital gains tax where ra is taxed at the rate τ and the proceeds are redistributed lump sum back to consumers

$$r = (1 - \tau)[f'(k) - \delta]$$

New Euler equation:

$$\theta \cdot \frac{\dot{\tilde{c}}}{\tilde{c}} = (1 - \tau)[f'(k) - \delta] - \rho - \theta g$$

so in steady state

$$f'(k^*) = \delta + \frac{\rho + \theta g}{1 - \tau}$$

Effects

Suppose economy is in steady state $(k^*, \tilde{c}^*)$ and tax declines to $\tau' < \tau$

What are the comparative dynamics?

— new steady-state equilibrium that is saddle-path stable $(k^{**}, \tilde{c}^{**})$
— since $\tau' < \tau \rightarrow k^{**} > k^*$
— equilibrium growth rate will still be g

Suppose change in tax is unanticipated and occurs at some date T

— at time T, $\dot{\tilde{c}} = 0$ curve shifts to the right $(k^{**} > k^*)$
— now previous steady state $\tilde{c}^*$ is above new stable arm
— consumption must drop immediately and then slowly increase along new stable arm