Chapter 2

Solow Model

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Robert M. Solow, 1924-

Robert M. Solow

Basic Solow model

First proposed by Robert Solow in 1956

- closed economy, with a single final good
- infinite time horizon indexed by $t \in (0, \infty)$

Economy is inhabited by a large number of agents who invest a constant exogenous fraction s of their income

Production function

All firms have access to the same aggregate production function

$$Y(t) = F\left[K(t), L(t), A(t)\right]$$

Capital K is used in production, and can be made one-for-one from the final good (putty-putty)

Technology A is assumed to be a public good

Assumptions

Assumption 1: Properties of the production function

The production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ is twice continuously differentiable in K and L, and satisfies

$$F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0 \qquad F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0$$
$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0 \quad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0$$

Moreover, F exhibits constant returns to scale in K and L (it is homogeneous of degree 1 in K and L)

Function homogeneity

Homogeneity: The function $g : \mathbb{R}^2 \to \mathbb{R}$ is **homogeneous of degree** m if and only if for all $(x, y) \in \mathbb{R}^2$ and $\lambda \in R_+$

$$g(\lambda x, \lambda y) = \lambda^m g(x, y)$$

Euler's theorem: Suppose that $g : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable in $x \in \mathbb{R}$ and $y \in \mathbb{R}$, with partial derivatives denoted by g_x and g_y and is homogeneous of degree m. Then

$$mg(x, y) = g_x(x, y)x + g_y(x, y)y$$
$$\forall\, x \in \mathbb{R} \text{ and } y \in \mathbb{R}$$

Moreover, $g_x(x, y)$ and $g_y(x, y)$ are themselves homogeneous of degree $m - 1$.

Homogeneity correspondence

If a function of homogeneous of degree m, it's partials are homogeneous of degree $m-1$:

$$\begin{aligned} & g(\lambda x, \lambda y) = \lambda^m g(x, y) \\ \frac{d}{dx} & \Rightarrow \lambda g_x(\lambda x, \lambda y) = \lambda^m g_x(x, y) \\ & \Rightarrow g_x(\lambda x, \lambda y) = \lambda^{m-1} g_x(x, y) \end{aligned}$$

and equivalently for y.

Similarly, deriving definition with respect to λ and evaluating at $\lambda = 1$ yields proof of Euler's theorem

Economic equilibrium

Markets are competitive: agents (firms and workers) act as if their choices do not affect prices

Labor market:

- agents inelastically supply total labor endowment $\bar{L}(t)$
- labor market clearing condition:

$$L(t) = \bar{L}(t)$$

- Assumption 1 and competitive labor markets guarantee $w(t) > 0$

Economic equilibrium

Capital market:

- agents own the capital stock of the economy and rent it to firms
- initial capital holdings $K(0)$
- capital market clearing condition: $K^s(t) = K^d(t)$
- rental price of capital is $R(t)$
- capital depreciates at rate δ
- net return faced by agents is $r(t) = R(t) - \delta$

Price of final good is $P(t) \rightarrow$ normalized to one in all periods

Firm optimization

Problem of a representative firm

$$\max_{L(t)\geq 0,\, K(t)\geq 0} F[K(t), L(t), A(t)] - w(t)L(t) - R(t)K(t)$$

Factor pricing equations, marginal products

$$\begin{aligned} w(t) &= F_L[K(t), L(t), A(t)] \\ R(t) &= F_K[K(t), L(t), A(t)] \end{aligned}$$

Note that at this stage it is best to think of this simply as a way of determining consistent prices. For general, w and R the optimum is not well defined.

Firm optimization

Proposition: *Zero profits for firms*

Suppose Assumption 1 holds. Then in the equilibrium of the Solow growth model, firms make no profits, and in particular,

$$Y(t) = w(t)L(t) + R(t)K(t)$$

Proof: Follows from Euler Theorem for the case of $m = 1$, i.e., constant returns to scale.

Assumptions

Assumption 2: *Inada conditions*

A function F satisfies the Inada conditions if

$$\begin{aligned}
\lim_{K \to 0} F_K(\cdot) = & \quad \infty \text{ and } \lim_{K \to \infty} F_K(\cdot) = 0 \text{ for all } L > 0 \\
\lim_{L \to 0} F_L(\cdot) = & \quad \infty \text{ and } \lim_{L \to \infty} F_L(\cdot) = 0 \text{ for all } K > 0
\end{aligned}$$

Moreover, $F(0, L, A) = F(K, 0, A) = 0$ for all K and L.

Assumption 2 is useful for existence of interior equilibrium

Dynamic equations

Law of motion capital stock

$$\dot{K}(t) = I(t) - \delta K(t)$$

National income accounting

$$Y(t) = C(t) + I(t)$$

Fundamental dynamic equation of Solow model

$$I(t) = sY(t)$$

$$\downarrow$$

$$\dot{K}(t) = sF\left[K(t), L(t), A(t)\right] - \delta K(t)$$

Equilibrium definition

Temporary assumptions:

- no population growth: $L(t) = L > 0$
- no technological progress: $A(t) = A$

Definition: *Equilibrium in Solow model*

For given L, A, and $K(0)$, an equilibrium is a set of functions $K(\cdot)$, $Y(\cdot)$, $C(\cdot)$, $w(\cdot)$, $R(\cdot)$ such that for all t:

- $K(t)$ satisfies the capital evolution equation
- $Y(t)$ is given by the production function
- $C(t)$ satisfies the agent's consumption equation
- $w(t)$ and $R(t)$ satisfy the factor pricing equations

Per capita variables

Capital-labor ratio: $k \equiv \frac{K}{L}$ (dropping t notation)

Output (income) per capita:

$$y \equiv \frac{Y}{L} = F\left[\frac{K}{L}, 1, A\right] \equiv f(k)$$

Per capita dynamic equation becomes:

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \frac{sF(K, L, A)}{K} - \delta = \frac{sf(k)}{k} - \delta$$

which implies

$$\dot{k} = sf(k) - \delta k$$

Factor prices

Using homogeneity we get:

$$R = F_K(K, L, A) = F_K\left(\frac{K}{L}, 1, A\right) = F_K(k, 1, A) = f'(k)$$

By constant returns to scale:

$$\begin{aligned} & F(K, L, A) = F_K(K, L, A)K + F_L(K, L, A)L \\ \Rightarrow\ & \frac{F(K, L, A)}{L} = F_K(K, L, A)\frac{K}{L} + F_L(K, L, A) \\ \Rightarrow\ & f(k) = f'(k)k + w \end{aligned}$$

Summarizing:

$$\begin{aligned} R &= f'(k) > 0 \\ w &= f(k) - kf'(k) > 0 \end{aligned}$$

Steady state

Definition: *Steady-state equilibrium*
A steady-state equilibrium without technological progress and population growth is an equilibrium path in which $k(t) = k^*$ for all t. Here k^* satisfies $sf(k^*) = \delta k^*$

Proposition: *Existence of steady state equilibrium*
Under Assumptions 1 and 2, there exists a unique steady state equilibrium in the Solow model where the capital-labor ratio $k^* \in (0, \infty)$ satisfies $sf(k^*) = \delta k^*$; per capita output is given by

$$y^* = f(k^*)$$

and per capita consumption is given by

$$c^* = (1 - s)f(k^*)$$

Comparative statics

Proposition: *Comparative statics Solow model*

Suppose Assumptions 1 and 2 hold and $f(k) = A\tilde{f}(k)$. Denote the steady-state level of the capital-labor ratio by $k^*(A, s, \delta)$ and the steady-state level of output by $y^*(A, s, \delta)$ when the underlying parameters are A, s and δ. Then we have

$$\frac{\partial k^*(\cdot)}{\partial A} > 0 \quad \frac{\partial k^*(\cdot)}{\partial s} > 0 \quad \frac{\partial k^*(\cdot)}{\partial \delta} < 0$$

$$\frac{\partial y^*(\cdot)}{\partial A} > 0 \quad \frac{\partial y^*(\cdot)}{\partial s} > 0 \quad \frac{\partial y^*(\cdot)}{\partial \delta} < 0$$

Golden rule determination

Comparative statics for c^*?

$$c^*(s) = (1-s)f(k^*(s)) = f(k^*(s)) - \delta k^*(s)$$

and differentiating with respect to s,

$$\frac{\partial c^*(s)}{\partial s} = [f'(k^*(s)) - \delta]\,\frac{\partial k^*}{\partial s}$$

Golden rule maximizes steady state consumption $\rightarrow s_{gold}$ is such that $\partial c^*(s_{gold})/\partial s = 0$, meaning

$$f'(k^*_{gold}) = \delta$$

Solow model with growth

Basic Solow model has no long-run growth, just transient equilibration dynamics when $k(0) \neq k^*$

Change assumptions:

- population growth: $\dot{L} = nL$
- technology growth: $\dot{A} = gA$

General production function:

$$Y = F(K, L, A)$$

Balanced growth

Does it admit a balanced growth path (BGP)?

$$Y(t) = Y(0)\exp(g_Y t)$$
$$C(t) = C(0)\exp(g_C t)$$
$$K(t) = K(0)\exp(g_K t)$$
$$I(t) = I(0)\exp(g_I t)$$

In particular, this means

$$\dot{Y} = g_Y Y$$
$$\dot{C} = g_C C$$
$$\dot{K} = g_K K$$
$$\dot{I} = g_I I$$

Solving BGP

From capital accumulation equation

$$\dot{K} = I - \delta K \quad \Rightarrow \quad g_K = \frac{\dot{K}}{K} = \frac{I}{K} - \delta$$

it must be $g_K = g_I$

From national accounts

$$C + I = Y$$

it must be $g_C = g_I = g_Y$

Is production function consistent with $g_Y = g_K$? and what is g_Y?

Technological change

Types of technological change for some constant returns to scale function $\tilde{F}$:

- Total factor (Hicks-neutral):

$$\tilde{F}(A, K, L) = AF(K, L)$$

- Capital augmenting (Solow-neutral):

$$\tilde{F}(A, K, L) = F(AK, L)$$

- Labor augmenting (Harrod-neutral):

$$\tilde{F}(A, K, L) = F(K, AL)$$

Could have changes in productivity of investment: $C + I/q = Y$ where q is productivity parameter

Kitchen sink growth

Consider all types of technological change together

$$Y = Z \cdot F(BK, AL)$$

where:

- $\dot{Z} = g_Z Z$ with $g_Z > 0$
- $\dot{B} = g_B B$ with $g_B > 0$
- $\dot{A} = g_A A$ with $g_A > 0$

Under the assumption of constant returns to scale Z is redundant

Special case

Proposition: *Cobb-Douglas production and BGP*

If F is a Cobb-Douglas production function of the type

$$Y = Z(BK)^{\alpha}(AL)^{1-\alpha}$$

then it is without loss of generality to focus on the case where $g_Z = g_B = 0$ and $g_A > 0$.

Proof: Define $\tilde{A} \equiv AB^{\frac{\alpha}{1-\alpha}} Z^{\frac{1}{1-\alpha}}$ and write

$$Y = K^{\alpha}(\tilde{A}L)^{1-\alpha}$$

noting that $g_{\tilde{A}} = g_A + \left(\frac{\alpha}{1-\alpha}\right) g_B + \left(\frac{1}{1-\alpha}\right) g_Z$

General case

Uzawa's Theorem: Consider the case of a standard Solow model with a production function $F(A, K, L)$ satisfying constant returns to scale and constant population growth n. If there exists $T < \infty$ such that for all $t \geq T$, $\dot{Y}(t)/Y(t) = g_Y > 0$, $\dot{K}(t)/K(t) = g_K > 0$, and $\dot{C}(t)/C(t) = g_C > 0$, then

1. $g_Y = g_K = g_C$
2. There exists $\tilde{F}$ that is homogeneous of degree 1 such that $\forall t \geq T$

$$Y(t) = \tilde{F}(K(t), \tilde{A}(t)L(t))$$

   and $g_{\tilde{A}}(t) = g_Y(t) - n$.

Interpretation: under our standard assumptions, it is without loss of generality to assume that technological change is purely labor augmenting.

Labor augmenting

Thus we will assume labor augmenting technological change

$$Y = F(K, AL)$$

Since function is constant returns to scale in capital K and effective labor AL:

$$\frac{Y}{AL} = F\left(\frac{K}{AL}, 1\right)$$

Partial converse to Uzawa's Theorem: BGP is technologically feasible for *any* constant returns to scale function F and

$$g_K = g_Y = g + n$$

Normalized variables

Assume production function of the type

$$Y = F(K, AL)$$

Define effective capital-labor ratio:

$$k \equiv \frac{K}{AL}$$

Output per-effective unit of labor:

$$y = \frac{Y}{AL} = F\left(\frac{K}{AL}, 1\right) \equiv f(k)$$

Along BGP y and k are constant, but there is growth in per-capita income (output) y

Dynamics

Key dynamic equation

$$
\begin{aligned}
\frac{\dot{k}}{k} &= \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} \\
&= \frac{sF(K, AL)}{K} - \delta - g - n \\
&= \frac{sf(k)}{k} - \delta - g - n
\end{aligned}
$$

which implies

$$\dot{k} = sf(k) - (\delta + g + n)k$$

Steady state

Steady state k^* is

$$(\delta + g + n)k^* = sf(k^*)$$

Proposition: *Steady state Solow model with growth*

Suppose Assumptions 1 and 2 hold. There exists a unique steady state equilibrium where the effective capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}$$

Per capita output and consumption grow at the rate g.

Differential equations

Question: given $k(0) > 0$, how does the economy behave along the transition path and does it tend to the steady state?

Math refresh 1: *Linear autonomous differential equations*

$$\dot{x}(t) = mx(t) + b$$

— steady state: $x^* = -b/m$
— general solution:

$$x^g(t) = -\frac{b}{m} + d\exp(mt)$$

— solution associated with boundary condition $x(0) = x_0$ is

$$x(t) = -\frac{b}{m} + \left(x_0 + \frac{b}{m}\right)\exp(mt)$$

— global asymptotically stability $\rightarrow x(t) \rightarrow x^*$ if $m < 0$

Nonlinearities

But Solow model is in general a non-linear differential equation

Math refresh 2: *Nonlinear autonomous differential equations*

$$\dot{x}(t) = g(x(t))$$

— steady state: $g(x^*) = 0$
— assume g is differentiable around x^*
— study local dynamics using

$$\dot{x}(t) \simeq g'(x^*)(x(t) - x^*)$$

— x^* is locally asymptotically stable if $g'(x^*) < 0$
— if $g(x(t)) < 0$ for all $x(t) > x^*$ and $g(x(t)) > 0$ for all $x(t) < x^*$, then x^* is globally asymptotically stable

Local stability

Consider the Cobb-Douglas case

$$\dot{k} = sk^{\alpha} - (\delta + g + n)k \equiv g(k)$$

Then we have

$$k^* = \left(\frac{s}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}$$

And the stability condition

$$g'(k) = s\alpha k^{\alpha-1} - (\delta + g + n)$$

which implies

$$g'(k^*) = -(1-\alpha)(\delta + g + n) < 0$$

Global stability

Solow dynamic equation continuous time

$$\dot{k}(t) = sf(k(t)) - (\delta + g + n)k(t) \equiv g(k(t))$$

Proposition: *Global stability Solow model continuous time*

Suppose Assumptions 1 and 2 hold, then the Solow growth model in continuous time is asymptotically stable, i.e., starting from any $k(0) > 0$, the effective capital-labor ratio converges to a steady-state value k^* or $k(t) \to k^*$.

Special case (Cobb-Douglas)

Dynamic equation:

$$\dot{k}(t) = sk(t)^\alpha - (\delta + g + n)k(t)$$

Change of variable: $x(t) \equiv k(t)^{1-\alpha}$

$$\dot{x}(t) = (1-\alpha)s - (1-\alpha)(\delta + g + n)x(t)$$

with solution

$$x(t) = \frac{s}{\delta + g + n} + \left[x(0) - \frac{s}{\delta + g + n}\right] \exp(-(1-\alpha)(\delta + g + n)t)$$

Starting from any $k(0)$:

$$k(t) \to k^* = (s/(\delta + g + n))^{1/(1-\alpha)}$$

Applications

Growth:

- exogenous growth: $F(K(t), L, A(t)) = A(t)K(t)^{\alpha}L^{1-\alpha}$ with $A(t)$ growing at rate g
- simplest endogenous growth $\rightarrow F(K(t)) = AK(t)$

Business cycles:

- $A(t)$ is stochastic
- jump process $\rightarrow$ two values: A^L and A^H
- Brownian motion $\rightarrow dA = \mu A + \sigma dW$