Prelim Solutions 2024

1 Eco-Malthus

(a) For the environmental resource, we find

$$g_E = \frac{\dot{E}}{E} = \gamma - \delta \cdot \frac{L}{E} = \gamma - \frac{\delta}{e}$$

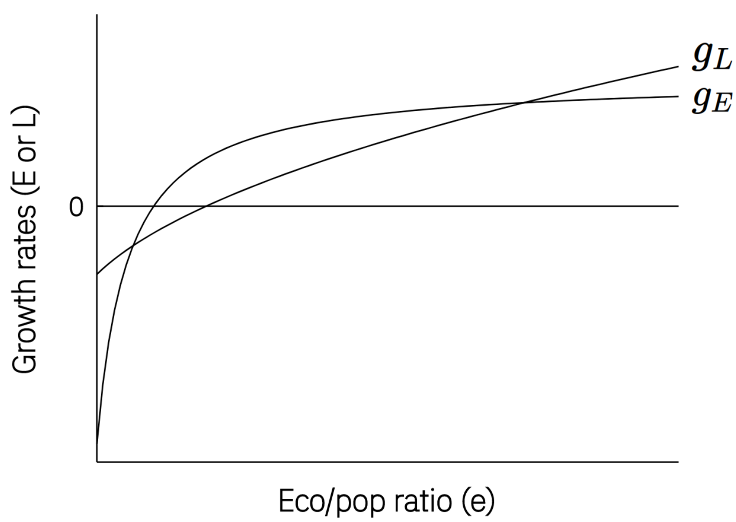
For population we find need to find the standard of living equation

$$y = \frac{Y}{L} = z\left(\frac{E}{L}\right)^{\alpha} = ze^{\alpha}$$

And then for population we find

$$g_L = \theta(y - \bar{y}) = \theta(ze^{\alpha} - \bar{y})$$

(b) Here we plot both g_E and g_L. Note that the intersections can have growth (y) values that are of any sign, though here we see the case where they straddle zero.



There are different cases depending on the level of overlap between the two growth curves. In the case above, there are two intersection points. The larger of the two (e_2^*) will be stable, while the smaller (e_1^*) will be unstable. Additionally, convergence to $e = 0$ is possible if the starting value is below e_1^*, otherwise we converge to e_2^*. In the interior steady state, e will converge to a constant (which in general can be positive or negative) and both the ecological resource (E) and the population (L) will grow at a common rate.

Another possible case is where there is no intersection between the curves (the single intersection case is measure zero). This is always possible for sufficiently large technology z or demographic parameter θ. In this case, population growth will always dominate resource growth and we will get $e \to 0$.

(c) Here we do the simple computations

$$0 = g_E(e_E^*) = \gamma - \frac{\delta}{e_E^*} \Rightarrow e_E^* = \frac{\delta}{\gamma}$$
$$0 = g_L(e_L^*) = \theta(z(e_L^*)^\alpha - \bar{y}) \Rightarrow e_L^* = \left(\frac{\bar{y}}{z}\right)^{\frac{1}{\alpha}}$$

One can see graphically that there is a positive intersection between the curves if (but not only if)

$$e_E^* < e_L^*$$
$$\frac{\delta}{\gamma} < \left(\frac{\bar{y}}{z}\right)^{\frac{1}{\alpha}}$$
$$z < \bar{y}\left(\frac{\gamma}{\delta}\right)^\alpha$$

since g_E is bounded above by γ, while g_L grows without bound as e rises.

(d) Looking at the graph above, an increase in technology will shift the g_L curve upwards. In the case that we are in a positive growth steady state (e_2^*), this will shift the e_2^* to the left (lower). The resulting growth rate will also be lower, as the g_E curve is upward sloping. Thus g_L will be lower, and hence y will be lower as well.

For a sufficiently large increase in z, we shift regimes into the no intersection case, and the steady state value of e will collapse to zero.

2 The Big One

(a) Using the standard approach, we find

$$\dot{a} + c = (r - n)a + w$$

Then the Hamiltonian is (with discount rate $\rho - n$)

$$H = u(c) + \mu \left[(r-n)a + w - c \right]$$

The conditions for optimality are

$$\begin{aligned} 0 = H_c &= u'(c) - \mu \\ (\rho - n)\mu - \dot{\mu} = H_a &= (r-n)\mu \end{aligned}$$

Combining these we find

$$\frac{\dot{c}}{c} = -\frac{\dot{\mu}}{\mu} = r - \rho$$

which together with the present value budget constraint characterizes the optimal consumption path.

(b) The factor prices should satisfy

$$\begin{aligned} R = F_K &= \alpha z \left(\frac{L}{K} \right)^{1-\alpha} = \alpha z k^{\alpha - 1} \\ w = F_L &= (1-\alpha) z \left(\frac{K}{L} \right)^{\alpha} = (1-\alpha) z k^{\alpha} \end{aligned}$$

The budget equation then simplifies to

$$\dot{k} + c = z k^{\alpha} - (\delta + n) k$$

And so the system of equations can be expressed as

$$\begin{aligned} \dot{k} &= z k^{\alpha} - (\delta + n) k - c \\ \dot{c} &= c \left[\alpha z k^{\alpha - 1} - (\delta + \rho) \right] \end{aligned}$$

(c) The phase diagram looks the same as in the notes, with the null-clines defined by

$$\begin{aligned} \dot{k} = 0 &\Leftrightarrow c = z k^{\alpha} - (\delta + n) k \\ \dot{c} = 0 &\Leftrightarrow k = \left(\frac{\alpha z}{\delta + \rho} \right)^{\frac{1}{1-\alpha}} = k^* \end{aligned}$$

So we have k^* above and c^* simplifies to

$$c^* = k^* \left[\frac{\delta + \rho}{\alpha} - (\delta + n) \right]$$

which is positive by virtue of $\rho - n > 0$.

(d) You can think about this case graphically through the phase diagram. Take the stable arm path (k, c) and project out to $(k/0.7, c)$. At discovery, we want to jump down from c^* so that we hit the projected stable arm after ten years. Then after the asteroid hits, we'll be exactly on the regular stable arm and will recover monotonically. Critically, there will be no discontinuity in c at impact, as the impact was anticipated. The only discontinuity will be at asteroid discovery, as we gained new information here and reoptimized.

The time plot for c will look like: drop at discovery, slow decline until impact, then exponential recovery. The time plot for k will look like: start buildup at discovery, large drop at impact, then exponential recovery.