Prelim 2024

1 Eco-Malthus

Imagine a society in which production is undertaken using both labor and some replenishable ecological resource. One prime example would be some kind of hunter-gatherer society. Let the ecological resource be doneted by $E > 0$, and suppose that it naturally replenishes at rate γ and is depleted by humans at rate δ, so that

$$\dot{E} = \gamma E - \delta L$$

The production function is then given according to

$$Y = zE^{\alpha}L^{1-\alpha}$$

Assume that the level of technology (z) is not growing. The population growth rate is classic Malthus with

$$g_L = \theta(y - \bar{y})$$

where $g_L = \dot{L}/L$ and $y = Y/L$.

(a) It makes sense to think about this system in terms of $e = E/L$. Derive expressions for g_E, g_L, and y purely in terms of e and model parameters.

(b) Plot g_E and g_L on the same graph as functions of e. The intersection points represent where $g_e = 0$. Describe the possible stable and unstable steady states of this model. What is the long-run behavior of population L and the standard of living y in these cases? [**Note**: there aren't closed form expressions for the steady state values of e, just depict these graphically.]

(c) Try to derive sufficient conditions for there to be a stable steady state with $e > 0$. It may be helpful to find the points where the growth curves cross zero, that is those satisfying $g_E(e^*_E) = 0$ and $g_L(e^*_L) = 0$.

(d) What is the effect of a one time increase in technology z? Specifically, how do the long-run values for e, g_L, and y change? Does the answer depend on the size of the change in z?

2 The Big One

Consider a Ramsey-style model where consumers have the utility function

$$U = \int_0^\infty \log(c(t))L(t)\exp(-\rho t)dt$$

and the population L grows at rate n. Consumers have access to a bond A that earns rate of return r, so the aggregate budget constraint is

$$\dot{A} + cL = rA + wL$$

Production is undertaken using capital and labor according to

$$Y = zK^\alpha L^{1-\alpha}$$

where technology z is fixed and capital depreciates at rate δ.

(a) Reformulate the budget constraint in per capita terms. Then write down the Hamiltonian optimization problem of the consumer and characterize the optimal consumption path.

(b) Find the factor prices R and w from the firm's optimality conditions. Using these, find the laws of motion for the joint (k, c) system.

(c) Draw a phase diagram depicting the dynamics of this system. Derive expressions for the steady state values of k and c.

(d) Astronomers discover an asteroid that will wipe out 30% of the capital stock (but no people) in 10 years. It will hit with certainty, Bruce Willis cannot save us. Use a new phase diagram to depict the outcome in this case. Additionally, draw time plots of k and c including the discovery, the impact, and the recovery.