Econ 3070: Midterm Solutions

1 Rise of the Machines

(a) The growth rate of capital is

$$\dot{k} = s_k y - (\delta + n)k$$

The growth rate of robots is

$$\dot{m} = s_m y - (\delta + n)m$$

The production function is

$$y = k^\alpha (1+m)^{1-\alpha}$$

(b) The growth rates of k and m can be expressed as

$$\begin{aligned}
\frac{\dot{k}}{k} &= s_k \left(\frac{m}{k}\right)^{1-\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - (\delta + n) \\
\frac{\dot{m}}{m} &= s_m \left(\frac{k}{m}\right)^{\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - (\delta + n)
\end{aligned}$$

Now let's calculate the growth rate of a

$$\begin{aligned}
\frac{\dot{a}}{a} &= \frac{\dot{k}}{k} - \frac{\dot{m}}{m} \\
&= s_k \left(\frac{m}{k}\right)^{1-\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - s_m \left(\frac{k}{m}\right)^{\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} \\
&= \left(\frac{k}{m}\right)^{\alpha-1} \left(\frac{1+m}{m}\right)^{1-\alpha} \left(s_k - \frac{k}{m} s_m\right) \\
&= a^{1-\alpha} b^{\alpha-1} (s_k - a s_m)
\end{aligned}$$

As for b, we find

$$\begin{aligned}
\frac{\dot{b}}{b} &= \frac{\dot{m}}{m(1+m)} = (1-b)\frac{\dot{m}}{m} \\
&= (1-b)\left[s_m a^\alpha b^{\alpha-1} - (\delta + n)\right]
\end{aligned}$$

In rate of change terms, we then have

$$\dot{a} = a^{\alpha} b^{\alpha-1}(s_k - a s_m)$$
$$\dot{b} = b(1-b)\left[s_m a^{\alpha} b^{\alpha-1} - (\delta + n)\right]$$

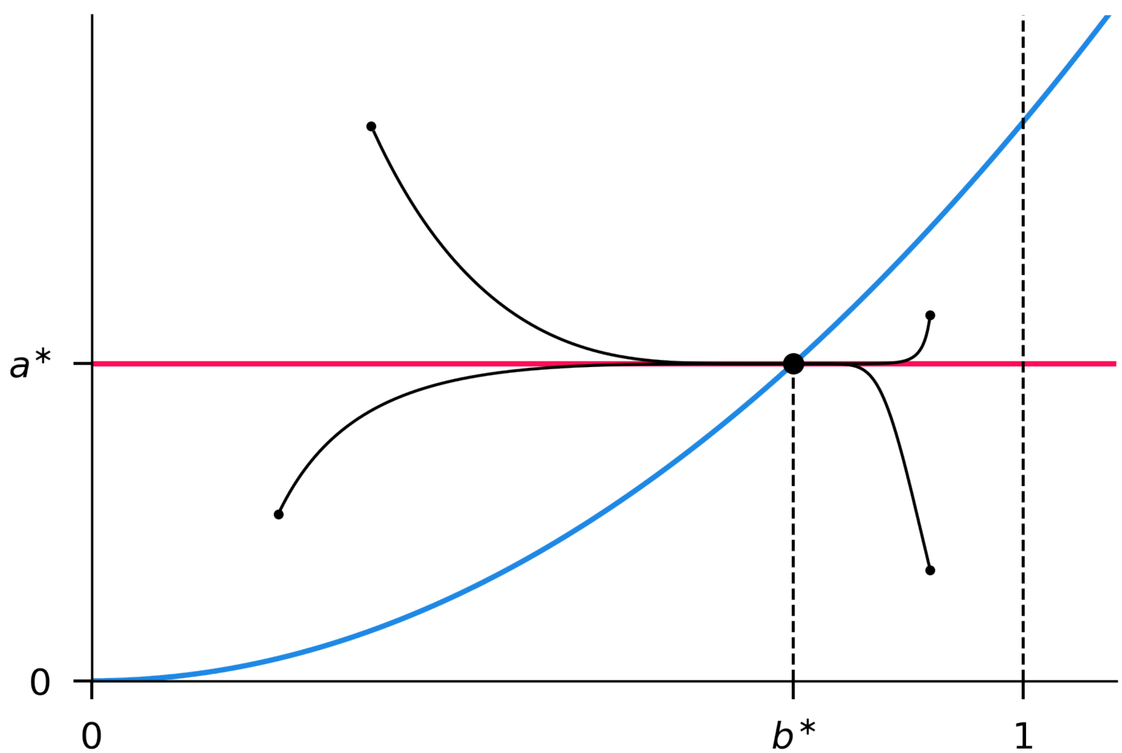
(c) For $\dot{a} = 0$ we can either have $a = 0$, $b = 0$ or

$$a = \frac{s_k}{s_m} \equiv a^*$$

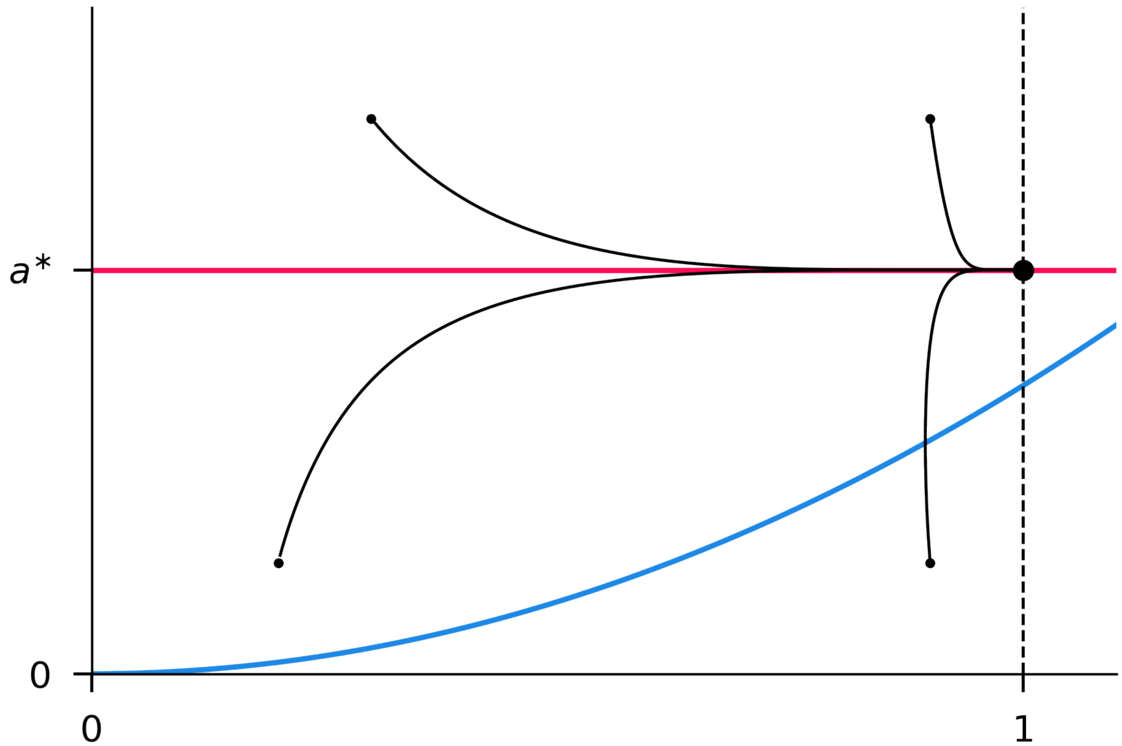
For $\dot{b} = 0$, we can have either $b = 0$, $b = 1$, or

$$a = \left(\frac{\delta + n}{s_m}\right)^{1/\alpha} b^{\frac{1-\alpha}{\alpha}}$$

Since I'm in less of a time crunch, I went ahead and computed the phase paths exactly. It's fine as long as your lines are going in the right direction. Notice that the steady state point is fully stable.



However, as thinking about part (d) might prompt you to realize, I've implicitly assumed that the nullclines cross in the space $b < 1$. If the crossing point happens at $b > 1$, we can't actually get there. Below is such a case



(d) There are two possibilities, as alluded to above. First, consider the crossing point of the nullclines. This occurs at $a = a^*$ and

$$b = \left(\frac{s_k^\alpha s_m^{1-\alpha}}{\delta + n}\right)^{\frac{1}{1-\alpha}} \equiv b^*$$

If $b^* < 1$, then this is our steady state. If $b^* \geq 1$, then in the limit we converge to $b = 1$ and $a = a^*$. In either case, the ratio of k to m converges to a^*. In the first case, we converge to

$$m^* = \frac{b^*}{1 - b^*} \quad \text{and} \quad k^* = \frac{s_k}{s_m} \cdot m^*$$

In the second case, m grows without bound, and hence so does k. We can go back to the growth rate equations to find

$$g_m = g_k = s_k^\alpha s_m^{1-\alpha} - (\delta + n)$$

which is positive if and only if $b^* > 1$.

Both of these steady states are stable when they exist. There are also some trivial steady states out there. This includes when $a = 0$ or $b = 0$, which correspond exactly to when $k = 0$ or $m = 0$. These are unstable in the same way the $k = 0$ is unstable in Solow. The case where $b = 1$ is not actually possible, since that would mean $m = \infty$.

2 Leisure Class

(a) I'm going to use B for consumers' bond holdings, since A is already technology. In the aggregate, the budget constraint is

$$C + \dot{B} = rB + wH$$
$$\Rightarrow c + \dot{b} = (r - n)b + wh$$

Dividing the production function by L, we get

$$\frac{Y}{L} = \left(\frac{K}{L}\right)^{\alpha} \left(\frac{AH}{L}\right)^{1-\alpha}$$
$$\Rightarrow y = k^{\alpha}(Ah)^{1-\alpha}$$

(b) Here we simply look at the marginal products

$$R = \frac{\partial Y}{\partial K} = \alpha K^{\alpha-1}(AH)^{1-\alpha} = \alpha \left(\frac{Ah}{k}\right)^{1-\alpha}$$
$$w = (1-\alpha)AK^{\alpha}(AH)^{-\alpha} = (1-\alpha)A\left(\frac{k}{Ah}\right)^{\alpha}$$

(c) The Hamiltonian in this case is simply

$$\hat{H} = u(c) - v(h) + \mu\left[(r-n)b + wh - c\right]$$

The first order condition on c yields

$$0 = \hat{H}_c = u'(c) - \mu \quad \Rightarrow \quad u'(c) = \mu \quad \Rightarrow \quad -\frac{\dot{\mu}}{\mu} = \varepsilon(c)\frac{\dot{c}}{c}$$

where

$$\varepsilon(c) = -\frac{u''(c)c}{u'(c)}$$

The FOC on h yields

$$0 = \hat{H}_h = -v'(h) + \mu w \quad \Rightarrow \quad v'(h) = \mu w$$

Combining the two FOCs yields

$$wu'(c) = v'(h)$$

The state evolution condition is

$$(\rho - n)\mu - \dot{\mu} = \hat{H}_b = \mu(r-n) \quad \Rightarrow \quad -\frac{\dot{\mu}}{\mu} = r - \rho$$

Combining these yields

$$\varepsilon(c)\frac{\dot{c}}{c} = r - \rho = \alpha\left(\frac{Ah}{k}\right)^{1-\alpha} - \delta - \rho$$

Imposing market clearing and using the budget constraint, we find

$$\begin{aligned} c + \dot{k} &= y - \delta k = k^{\alpha}(Ah)^{1-\alpha} - \delta k \\ \Rightarrow \dot{k} &= k^{\alpha}(Ah)^{1-\alpha} - \delta k - c \end{aligned}$$

In summary, the evolution of c, h, and k is described by

$$\begin{aligned} \varepsilon(c)\frac{\dot{c}}{c} &= \alpha\left(\frac{Ah}{k}\right)^{1-\alpha} - \delta - \rho \\ \dot{k} &= k^{\alpha}(Ah)^{1-\alpha} - \delta k - c \\ v'(h) &= (1-\alpha)A\left(\frac{k}{Ah}\right)^{\alpha} u'(c) \end{aligned}$$

(d) In this case, the above system of equations becomes

$$\begin{aligned} &(1) \quad \theta g_c = \alpha\left(\frac{Ah}{k}\right)^{1-\alpha} - \delta - \rho \\ &(2) \quad \dot{k} = k^{\alpha}(Ah)^{1-\alpha} - \delta k - c \\ &(3) \quad h^{\eta} = (1-\alpha)A\left(\frac{k}{Ah}\right)^{\alpha} c^{-\theta} \end{aligned}$$

The production function in growth rate terms implies

$$g_y = \alpha g_k + (1-\alpha)(g + g_h)$$

while g_c being constant in (1) requires

$$g_k = g + g_h$$

which along with equation (2) implies

$$g_c = g_y = g_k = g + g_h$$

Now consider equation (3) in growth rate form

$$\begin{aligned} \eta g_h &= g - \theta g_c = g - \theta(g + g_h) \\ \Rightarrow \quad g_h &= \left(\frac{1-\theta}{\eta+\theta}\right) g \end{aligned}$$

Finally, this implies

$$g_c = g_k = \left(\frac{\eta + 1}{\eta + \theta}\right) g$$

So we find that there is always positive growth in consumption and capital, but whether hours are growing or shrinking depends on whether $\theta < 1$ or $\theta > 1$. And in the logarithmic case where $\theta = 1$, we get that hours are actually constant and consumption and capital grow at rate g.

This is beyond the scope of the question, but in the $\theta = 1$ case, you can actually solve through to find

$$\begin{aligned}
\frac{Ah}{k} &= \left(\frac{\rho + \delta + g}{\alpha}\right)^{\frac{1}{1-\alpha}} \\
\frac{c}{k} &= \frac{\rho + (1-\alpha)(\delta + g)}{\alpha} \\
h &= \left[\frac{(1-\alpha)\rho + (1-\alpha)(\delta + g)}{\rho + (1-\alpha)(\delta + g)}\right]^{\frac{1}{1+\eta}} < 1
\end{aligned}$$