Econ 3070: Midterm Solutions

1 Tommy Malthus Does the Blues

(a) In this case we have

$$g_L = \frac{\dot{L}}{L} = b\left(\frac{Y}{L} - c\right) - d$$
$$= b\left[y - \left(c + \frac{d}{b}\right)\right]$$

Thus we can see that b in this model corresponds to θ in the original Malthusian model. The relationship to $\bar{y}$ is a bit less clear, as c is endogenous. We can at least establish that if $y < \frac{d}{b}$ then g_L must be negative, since c cannot be negative. If c were fixed, then we could cleanly identify $\bar{y}$ with $c + \frac{d}{b}$.

As for interpretation, $1/b$ represents the resource cost of rearing one child. And so $\frac{d}{b} = d \cdot \frac{1}{b}$ is the resource cost of rearing enough children to counteract the death rate.

(b) Since L is endogenous, we can't normalize it away before optimizing, so it must be a state variable. The Hamiltonian here is

$$H = u(c)L + \mu\left[b(zK^{\alpha}L^{1-\alpha} - cL) - dL\right]$$

It's also useful to note that

$$y = \frac{Y}{L} = z\left(\frac{K}{L}\right)^{\alpha}$$

(c) The first order condition for c yields a familiar condition

$$0 = H_c = u'(c)L + \mu bL \quad \Rightarrow \quad u'(c) = b\mu \quad \Rightarrow \quad \theta\frac{\dot{c}}{c} = -\frac{\dot{\mu}}{\mu}$$

The state multiplier equation yields

$$\rho\mu - \dot{\mu} = H_L = u(c) + \mu\left[b(1-\alpha)y - (bc + d)\right]$$

We can express this in the form

$$-\frac{\dot{\mu}}{\mu} = \frac{bc}{1-\theta} + (1-\alpha)by - (bc + d + \rho)$$

Combining these equations, we find

$$\theta\frac{\dot{c}}{c} = \frac{bc}{1-\theta} + (1-\alpha)by - (bc+d+\rho)$$
$$\frac{\dot{L}}{L} = by - (bc+d)$$

It's actually a bit cleaner to reformulate the system in terms of (c, y) since $g_y = -\alpha g_L$, so that

$$\theta\frac{\dot{c}}{c} = \frac{bc}{1-\theta} + (1-\alpha)by - (bc+d+\rho)$$
$$\frac{1}{\alpha}\frac{\dot{y}}{y} = (bc+d) - by$$

(d) This is still Malthus, so we will converge to some fixed level of population. Using the $\dot{L} = 0$ condition, we find

$$by = bc + d$$

Using this and the $\dot{c} = 0$ condition, we find

$$\frac{bc}{1-\theta} = \alpha by + \rho = \alpha(bc+d) + \rho$$

Solving this for c, we find

$$bc^* = (1-\theta)\left[\frac{\alpha d + \rho}{1-\alpha(1-\theta)}\right]$$

Plugging this back in we can find

$$by^* = \frac{d + \rho(1-\theta)}{1-\alpha(1-\theta)}$$

Finally, we can back out the population with

$$L^* = K\left(\frac{z}{y^*}\right)^{\frac{1}{\alpha}}$$

The assumptions required for c^* to be both positive and bounded are

$$\alpha(1-\theta) < 1 \quad \text{and} \quad \theta < 1$$

which are guaranteed to be true whenever $\theta \in (0, 1)$.

2 Something's Up With The Machines

(a) We can see that the production function will becomes

$$y = \left[m^{\varepsilon}k^{1-\varepsilon} + 1\right]^{\frac{1}{1-\varepsilon}}$$

The aggregate budget constraint is

$$\dot{B} = rB + wL - cL$$

In normalized terms, this becomes

$$\dot{b} = (r - n)b + w - c$$

(b) The consumer side here is standard and will yield the Euler equation

$$\frac{\dot{c}}{c} = r - \rho = R - (\rho + \delta)$$

On the firm side, we find

$$w = \frac{\partial Y}{\partial L} = Y^{\varepsilon}L^{-\varepsilon} = y^{\varepsilon}$$
$$R = \frac{\partial Y}{\partial K} = Y^{\varepsilon}m^{\varepsilon}K^{-\varepsilon} = \left(\frac{my}{k}\right)^{\varepsilon}$$

Note that this implies $1 = w + Rk$. Plugging this into the budget equation and noting $b = k$, we get

$$\dot{k} = y - (\delta + n)k - c$$

(c) In steady state, we'll have

$$\rho + \delta = R = \left(\frac{my}{k}\right)^{\varepsilon} = \left[m + \left(\frac{m}{k}\right)^{1-\varepsilon}\right]^{\frac{\varepsilon}{1-\varepsilon}}$$

Solving this yields

$$k^* = m\left[\frac{1}{(\rho + \delta)^{\frac{1-\varepsilon}{\varepsilon}} - m}\right]^{\frac{1}{1-\varepsilon}}$$

So the requirement is that

$$m < (\delta + \rho)^{\frac{1-\varepsilon}{\varepsilon}}$$

We can also find consumption

$$
\begin{aligned}
c = y - (\delta+n)k &= k\left[\frac{y}{k} - (\delta+n)\right] \\
&= k\left[\frac{(\delta+\rho)^{\frac{1}{\varepsilon}}}{m} - (\delta+n)\right] \\
&= k(\delta+\rho)\left[\frac{(\delta+\rho)^{\frac{1-\varepsilon}{\varepsilon}}}{m} - \frac{\delta+n}{\delta+\rho}\right] > 0
\end{aligned}
$$

where here we use $\rho > n$.

(d) We may have growth in per capita terms here. Consider the case where $\varepsilon \in (0,1)$

$$
\begin{aligned}
g_c &= \left[m + \left(\frac{m}{k}\right)^{1-\varepsilon}\right]^{\frac{\varepsilon}{1-\varepsilon}} - (\delta+\rho) \\
&\to m^{\frac{\varepsilon}{1-\varepsilon}} - (\delta+\rho)
\end{aligned}
$$

And for capital

$$g_k = \frac{y}{k} - (\delta+n) - \frac{c}{k}$$

So in the limit we must have

$$g_y = g_k = g_c = m^{\frac{\varepsilon}{1-\varepsilon}} - (\delta+\rho)$$

In the limit, we get

$$y = m^{\frac{\varepsilon}{1-\varepsilon}} k \quad \Rightarrow \quad \frac{y}{k} = m^{\frac{\varepsilon}{1-\varepsilon}}$$

Solving for the c/k ratio

$$
\begin{aligned}
\frac{c}{k} &= \frac{y}{k} - (\delta+n) - g_k \\
&= m^{\frac{\varepsilon}{1-\varepsilon}} - (\delta+n) - m^{\frac{\varepsilon}{1-\varepsilon}} + (\delta+\rho) \\
&= \rho - n
\end{aligned}
$$

And the consumption rate is

$$\frac{c}{y} = \frac{c}{k}\frac{k}{y} = \frac{\rho-n}{m^{\frac{\varepsilon}{1-\varepsilon}}}$$

which is guaranteed to be less then one under the parametric condition from the previous part. When $\varepsilon > 1$, we find instead that

$$g_y = g_k = g_c = -(\delta+\rho)$$

and

$$\frac{c}{k} = m^{\frac{\varepsilon}{1-\varepsilon}} + (\rho - n) \quad \text{and} \quad \frac{c}{y} = 1 - \frac{\rho-n}{m^{\frac{\varepsilon}{1-\varepsilon}}}$$

In this case, we require $\rho - n < m^{\frac{\varepsilon}{1-\varepsilon}}$, which is not equivalent to the condition from the previous part (because $\varepsilon > 1$).