Midterm Review Solutions

1 Ramsey Revisited

(a) From the production function we can see

$$g_Y = \alpha g_K + \beta(g + n)$$

From the capital accumulation and investment equation, we get

$$g_Y = g_C = g_I = g_K$$

Thus we find

$$g_K = \frac{\beta}{1-\alpha}(g + n)$$

This suggests we should normalize growing variables by

$$x = (AL)^{\frac{\beta}{1-\alpha}}$$

We will denote variables normalized by x with lower case letters.

(b) The aggregate budget constraint is

$$cL + \dot{B} = wL + rB + \Pi$$

First note that

$$\frac{\dot{b}}{b} = \frac{\dot{B}}{B} - \hat{g} \quad \Rightarrow \quad \frac{\dot{B}}{x} = \dot{b} + \hat{g}b$$

where

$$\hat{g} \equiv g_x = \frac{\beta}{1-\alpha}(g + n)$$

Denoting variables x/L with a tilde, we get

$$\tilde{c} + \dot{b} = \tilde{w} + (r - \hat{g})b + \pi$$

The production function reduces simply to

$$y = k^\alpha$$

(c) First, we can express the utility function as

$$\hat{U} = \int_0^\infty \log(\tilde{c}) \exp(-(\rho - n)t) dt \; + \; \text{Constant}$$

The commune's Hamiltonian is

$$\hat{H} = \log(\tilde{c}) + \mu[\tilde{w} + (r - \hat{g})b + \pi - \tilde{c}]$$

The first order condition is

$$\frac{1}{\tilde{c}} = \mu \quad \Rightarrow \quad -\frac{\dot{\mu}}{\mu} = \frac{\dot{\tilde{c}}}{\tilde{c}}$$

The costate condition is

$$\begin{aligned} (\rho - n)\mu - \dot{\mu} &= \mu(r - \hat{g}) \\ \Rightarrow \; -\frac{\dot{\mu}}{\mu} &= r - \rho + n - \hat{g} \end{aligned}$$

Combining these yields the Euler equation

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = r - \rho + n - \hat{g}$$

(d) From the firm optimality conditions we get

$$R = \alpha K^{\alpha-1}(AL)^\beta \quad \Rightarrow \quad Rk = \alpha k^\alpha$$

and

$$w = A\beta K^\alpha (AL)^{\beta-1} \quad \Rightarrow \quad \tilde{w} = \beta k^\alpha$$

Thus we have

$$\begin{aligned} \Pi &= K^\alpha (AL)^\beta - RK - wL \\ \Rightarrow \pi &= k^\alpha - Rk - \tilde{w} = (1 - \alpha - \beta)k^\alpha \end{aligned}$$

Whatever the case, we also have

$$\begin{aligned} \pi + Rk + \tilde{w} &= k^\alpha \\ \Rightarrow \pi + rk + \tilde{w} &= k^\alpha - \delta k \end{aligned}$$

Turning to the budget equation and imposing $B = K$, and hence $b = k$, we get

$$\tilde{c} + \dot{k} = \tilde{w} + (r - \hat{g})b + \pi = k^\alpha - (\delta + \hat{g})k$$

The law of motion is then

$$\dot{k} = k^\alpha - (\delta + \hat{g})k - \tilde{c}$$

In steady state we have

$$\alpha k^{\alpha-1} = R = \rho - n + \delta + \hat{g}$$

so that

$$k^* = \left(\frac{\alpha}{\rho - n + \delta + \hat{g}}\right)^{\frac{1}{1-\alpha}}$$

and finally

$$\begin{aligned} c^* &= (k^*)^\alpha - (\delta + \hat{g})k^* \\ &= \frac{k^*}{\alpha}\left[(\rho - n) + (1-\alpha)(\delta + \hat{g})\right] \end{aligned}$$

2 Capital Utilization

(a) Here the utility function can be expressed as

$$U = \int_0^\infty \log(c)\exp(-(\rho - n)t)dt$$

Normalized production is given by

$$y = xk^\alpha$$

and the law of motion for capital is

$$\dot{k} = i - (\delta(x) + n)k$$

The Hamiltonian is then

$$H = u(xk^\alpha - i) + \mu(i - (\delta(x) + n)k)$$

(b) Optimality here implies

$$H_i = 0 \quad \Rightarrow \quad u'(c) = \mu$$

and

$$H_x = 0 \quad \Rightarrow \quad u'(c)k^\alpha = \mu k \delta'(x)$$

and

$$(\rho - n)\mu - \dot{\mu} = H_k \quad \Rightarrow \quad (\rho + \delta(x))\mu - \dot{\mu} = x\alpha k^{\alpha-1} u'(c)$$

Simplifying

$$\frac{\dot{c}}{c} = -\frac{\dot{\mu}}{\mu} = x\alpha k^{\alpha-1} - \rho - \delta(x)$$

and

$$\delta'(x) = k^{\alpha-1}$$

(c) In steady state

$$\rho + \delta(x) = xk^{\alpha-1} = \alpha x \delta'(x)$$

which implies

$$\frac{\rho + \delta(x)}{\delta(x)} = \alpha \frac{x\delta'(x)}{\delta(x)} = \alpha\varepsilon_\delta$$
$$\Rightarrow \delta(x) = \frac{\rho}{\alpha\varepsilon_\delta - 1}$$

Then

$$k = \left[\frac{1}{\delta'(x)}\right]^{\frac{1}{1-\alpha}}$$

and

$$c = xk^\alpha - \delta(x)k = k[xk^{\alpha-1} - \delta(x)]$$
$$= k[x\delta'(x) - \delta(x)] = (\varepsilon_\delta - 1)\delta(x)k$$

(d) We require $\alpha\varepsilon_\delta > 1$ for the existence of a steady state.

For the transition, low capital means high intensity x. What about y? We can write

$$y = x\left[\frac{1}{\delta'(x)}\right]^{\frac{\alpha}{1-\alpha}} \sim x^{1-(\varepsilon_\delta - 1)\frac{\alpha}{1-\alpha}} = x^{\frac{1-\alpha\varepsilon_\delta}{1-\alpha}}$$

So then as long as a steady state exists, the level out output will be below steady state and converge monotonically.