Midterm Review

Both problems feature Ramsey-type models in which there is a representative commune with utility function

$$U = \int_0^\infty \log(c(t))L(t)\exp(-\rho t)dt$$

where the population L grows at rate n. Capital (K) and labor (L) are used in production of final good (Y). Final goods can be invested one-for-one into capital creation (I).

1 Ramsey Revisited

Consider a Ramsey-type model but with decreasing returns to scale in production. In particular, in the aggregate we will have

$$\begin{aligned} Y &= K^\alpha (AL)^\beta \\ \dot{K} &= I - \delta K \end{aligned}$$

with $\alpha + \beta \leq 1$. Technology (A) grows at rate g and capital depreciates at rate δ. Assets (B) earn a rate of return r and workers earn wage w. The firm gives profits Π back to the commune.

(a) Find the steady state growth rates of aggregate output and capital required for a BGP in this model. What does this imply about how we should normalize growing variables?

(b) Write down the budget constraint of the representative commune. Convert this into a normalized equation featuring only variables that do not grow in the long run. Do the same for the production function.

(c) Write down the Hamiltonian for the commune's consumption-savings problem and derive their Euler equation characterizing the growth rate of normalized consumption.

(d) Find factor prices R and w and use these to derive a law of motion for normalized capital. Characterize the steady state allocation of this model, being as explicit as possible.

2 Capital Utilization

Now consider a Ramsey-type model where the producer can use capital more or less intensely. When used more intensely, it generates more output but depreciates faster. In particular, for a chosen level of intensity x, capital depreciates at rate $\delta(x)$. Thus in the aggregate we have

$$Y = xK^{\alpha}L^{1-\alpha}$$
$$\dot{K} = I - \delta(x)K$$

where $\delta(0) = 0$, $\delta'(0) = 0$, $\delta''(\cdot) > 0$, and $\lim_{x\to\infty} \delta(x) = \infty$.

(a) Formulate the Hamiltonian for the social planner's problem in per capita terms and describe what the respective choice and state variables are. [Note that x can be chosen freely at each instant.]

(b) Derive conditions characterizing the optimal path of investment i, capital k, utilization intensity x.

(c) Supposing that the depreciation function $\delta(\cdot)$ has constant elasticity ε_δ (that is $\frac{x\delta'(x)}{\delta(x)} = \varepsilon_\delta$ for all x), find the steady state values for $\delta(x)$, k, and c.

(d) What conditions are required for the existence of a steady state? Under these conditions, supposing we start from a capital level below steady state, what do the transition paths for x, k, and y look like?