Econ 3070: Final Exam Solutions

1 Research Subsidy

(a) This mirrors the derivation in the notes. The final goods producer will have inverse demand function

$$p_i = \frac{y}{y_i}$$

Seeing this, the intermediate monopolist will employ a limit pricing strategy, meaning

$$p_i = \frac{w}{q_{-i}} = \frac{\lambda w}{q_i}$$

This yields production outcomes

$$y_i = \frac{q_i y}{\lambda w} \quad \Rightarrow \quad \ell_i = \frac{y}{\lambda w}$$

Labor market clearing implies that

$$P = \int_0^1 \ell_i di = \frac{y}{\lambda w} \quad \Rightarrow \quad w = \frac{y}{\lambda P}$$

Plugging this back into the production outcome we get

$$y_i = q_i P \quad \text{and} \quad \ell_i = P$$

Aggregating this up, we then find

$$y = QP \quad \text{where} \quad \log(Q) = \int_0^1 \log(q_i) di$$

Finally, we can compute the desired outcomes as

$$\pi_i = (1 - \lambda^{-1}) QP \quad \text{and} \quad w = \frac{Q}{\lambda}$$

(b) Using the usual methods, we can express v_i as

$$v_i = \frac{\pi_i}{r + \tau - g_v}$$

In this case, we have $r - g_v = r - g_c = \rho$, so that

$$v_i = \frac{\pi_i}{\rho + \tau}$$

Now the research subsidy finally comes in at the free entry condition, as it modifies the net cost of labor

$$\begin{aligned}\gamma v_i &= (1-s)w \\ \frac{\gamma(1-\lambda^{-1})QP}{\rho+\tau} &= \frac{(1-s)Q}{\lambda} \\ \frac{\gamma(\lambda-1)}{\rho+\gamma R} &= \frac{1-s}{1-R}\end{aligned}$$

Solving this yields

$$R^* = \frac{(\lambda-1)-(1-s)(\rho/\gamma)}{\lambda-s}$$

From here we can find

$$\tau^* = \frac{(\lambda-1)\gamma-(1-s)\rho}{\lambda-s}$$

and finally $g^* = \log(\lambda)\tau^*$

(c) Let's assume that $\ell_i = P$, as we showed in class. Then the Hamiltonian of the social planner is

$$H = \log(Q(1-R)) + \mu\log(\lambda)\gamma RQ$$

The first order condition yields

$$0 = H_R = -\frac{1}{1-R} + \mu\log(\lambda)\gamma Q$$

The state evolution equation yields

$$\rho\mu - \dot{\mu} = H_Q = \frac{1}{Q} + \mu\log(\lambda)\gamma R = \mu\log(\lambda)\gamma$$

Combining these, we find

$$\log(\lambda)\gamma - \rho = -\frac{\dot{\mu}}{\mu} = g = \log(\lambda)\gamma R$$

And so we arrive at

$$\hat{R} = 1 - \frac{\rho}{\log(\lambda)\gamma}$$

This leads to

$$\hat{\tau} = \gamma - \frac{\rho}{\log(\lambda)} \quad \text{and} \quad \hat{g} = \log(\lambda)\gamma - \rho$$

(d) This amounts to solving the equation $R^*(s) = \hat{R}$

$$\begin{aligned}
\frac{(\lambda-1)-(1-s)(\rho/\gamma)}{\lambda - s} &= 1 - \frac{\rho}{\log(\lambda)\gamma} \\
\frac{(1-s)(\rho/\gamma+1)}{\lambda - s} &= \frac{\rho}{\log(\lambda)\gamma} \\
\log(\lambda)(1-s)(\rho+\gamma) &= \rho(\lambda - s)
\end{aligned}$$

This leads to the solution

$$\hat{s} = \frac{\log(\lambda)(\rho+\gamma) - \lambda\rho}{\log(\lambda)(\rho+\gamma) - \rho}$$

Note that this only really makes sense when $\hat{R} > 0$, which means

$$\log(\lambda)\gamma \geq \rho \quad \Rightarrow \quad \log(\lambda)(\rho+\gamma) - \rho > 0$$

When the above condition does not hold, the optimal level is zero and we should just tax research as much as possible. Otherwise the $\hat{s}$ equation works. Consider the case where the above condition holds with equality

$$\hat{s}_0 = \frac{\log(\lambda) - (\lambda - 1)}{\log(\lambda)\rho} < 0$$

This reflects the fact that at low values of λ, there is too much research in equilibrium and we should tax it. We can also see that $\hat{s}$ is positive whenever

$$H(\lambda) \equiv \frac{\log(\lambda)}{\lambda} > \frac{\rho}{\rho+\gamma}$$

One can show that H achieves a maximum at $\lambda = e$, with value of $1/e$. So then there exists a λ for which $\hat{s}$ is positive if and only if

$$(e-1)\rho < \gamma$$

Finally, as λ grows very large, $\hat{s}$ becomes arbitrarily negative, reflecting the fact that there is overinvestment in equilibrium.

2 Researcher Skill

The number of new product created is just the integral of their flow probabilities

$$\begin{aligned}\dot{N} &= \int_0^R \gamma(j)dj \\ &= \int_0^R \bar{\gamma}\eta N^{\phi} j^{\eta-1} dj \\ &= \bar{\gamma} N^{\phi} R^{\eta}\end{aligned}$$

Following Jones, in growth rate terms, we find

$$g = \frac{\dot{N}}{N} = \bar{\gamma} N^{\phi-1} R^{\eta} = \frac{\bar{\gamma} R^{\eta}}{N^{1-\phi}}$$

Thus to achieve a constant growth rate, we should have

$$\eta n = (1-\phi) g \quad \Rightarrow \quad g^* = \frac{\eta n}{1-\phi}$$

(b) To maximize their profit, they will equate marginal product with price for each good, meaning

$$p_i = \frac{\partial y}{\partial y_i} = \left(\frac{y}{y_i}\right)^{\frac{1}{\varepsilon}} \quad \Rightarrow \quad y_i = p_i^{-\varepsilon} y$$

In response to a constant elasticity demand function, we know that the intermediate monopolist will charge a constant markup over cost that is a function of that elasticity, namely

$$p_i = \left(\frac{\varepsilon}{\varepsilon - 1}\right) w$$

(c) Because price is invariant to i, so are y_i and ℓ_i. Thus by labor market clearing, we have

$$y_i = \ell_i = \frac{P}{N}$$

Plugging this into the final good aggregator, we find

$$y = N^{\frac{\varepsilon}{\varepsilon-1}} \left(\frac{P}{N}\right) = N^{\frac{1}{\varepsilon-1}} P$$

The demand function now tells us

$$\ell_i = y_i = p_i^{-\varepsilon} y = \left[\left(\frac{\varepsilon-1}{\varepsilon}\right)\frac{1}{w}\right]^{\varepsilon} N^{\frac{1}{\varepsilon-1}} P$$

Imposing labor market clearing, we can find the wage

$$P = \left[\left(\frac{\varepsilon-1}{\varepsilon}\right)\frac{1}{w}\right]^{\varepsilon} N^{\frac{\varepsilon}{\varepsilon-1}} P \quad \Rightarrow \quad w = \left(\frac{\varepsilon-1}{\varepsilon}\right) N^{\frac{1}{\varepsilon-1}}$$

which also gives us the actual goods price $p_i = N^{\frac{1}{\varepsilon-1}}$. Combining all these, we arrive at the profit

$$\begin{aligned}
\pi_i &= p_i y_i - w\ell_i \\
&= N^{\frac{1}{\varepsilon-1}} \frac{P}{N} - \left(\frac{\varepsilon-1}{\varepsilon}\right) N^{\frac{1}{\varepsilon-1}} \frac{P}{N} \\
&= \frac{1}{\varepsilon} N^{\frac{1}{\varepsilon-1}} \frac{P}{N} = \frac{w}{\varepsilon-1}\frac{P}{N}
\end{aligned}$$

(d) Again, we can express the value as

$$v_i = \frac{\pi_i}{r - g_v}$$

Here we have $g_v = g_c - g$, so we arrive at

$$\bar{v} = v_i = \frac{\pi_i}{\rho + g}$$

For our free entry condition, we consider the worker R who is just indifferent between being a researcher and getting $\bar{v}$ with probability $\gamma(R)$ and being a worker that earns wage w. Any worker $j < R$ will strictly prefer research and and worker $j > R$ will strictly prefer production. Thus this R will also represent the total number of research workers. And so the indifference condition is

$$\begin{aligned}
\gamma(R)\bar{v} &= w \\
\eta\bar{\gamma}N^{\phi}R^{\eta-1}\bar{v} &= w \\
\eta\bar{\gamma}N^{\phi-1}R^{\eta-1}\frac{1}{\rho+g}\frac{wP}{\varepsilon-1} &= w \\
\eta\frac{\bar{\gamma}R^{\eta}}{N^{1-\phi}}\frac{1}{\rho+g}\frac{1}{\varepsilon-1} &= \frac{R}{P} \\
\frac{\eta}{\varepsilon-1}\frac{g}{\rho+g} &= \frac{R}{P}
\end{aligned}$$

Letting $s_R = \frac{R}{L}$, we can write this as

$$\frac{\eta}{\varepsilon-1}\frac{g}{\rho+g} = \frac{s_R}{1-s_R}$$

Solving this yields

$$s_R^* = \frac{\eta g^*}{\eta g^* + (\varepsilon - 1)(\rho + g^*)}$$

where

$$g^* = \frac{\eta n}{1 - \phi}$$