Solutions: PS3

1 Targeted Innovation

(a) Here the optimization problem of the final good aggregator yields the inverse demand function

$$p_i = \frac{\partial y}{\partial y_i} = \frac{\alpha_i}{y_i} y$$

Limit pricing dictates that we set price to the marginal cost of the competitor

$$p_i = \frac{w\lambda}{q_i} \quad \Rightarrow \quad y_i = \frac{\alpha_i q_i}{w\lambda} y$$

Thus labor utilization is

$$w\ell_i = \frac{\alpha_i}{\lambda} y \quad \Rightarrow \quad \pi_i = \alpha_i \left(\frac{\lambda - 1}{\lambda}\right) y$$

Labor market clearing then implies

$$wP = \frac{\mathbb{E}[\alpha_i]}{\lambda} y = \frac{1}{\lambda} y \quad \Rightarrow \quad w = \frac{1}{\lambda} \frac{y}{P}$$

(b) The value of a product line is

$$rv_i - \dot{v}_i = \pi_i - \tau_i v_i \quad \Rightarrow \quad v_i = \frac{\pi_i}{\rho + \tau_i}$$

The free entry in this case is

$$\begin{aligned}
& \gamma v_i = w \\
\Rightarrow \quad & \frac{\gamma \alpha_i}{\rho + \tau_i} \frac{\lambda - 1}{\lambda} y = \frac{1}{\lambda} \frac{y}{P} \\
\Rightarrow \quad & \gamma(\lambda - 1)\alpha_i(1 - R) = \rho + \gamma R_i \\
\Rightarrow \quad & \gamma(\lambda - 1)(1 - R) = \rho + \gamma R \\
\Rightarrow \quad & R^* = \frac{(\lambda - 1) - \frac{\rho}{\gamma}}{\lambda}
\end{aligned}$$

Mapping this back into individual products, we find

$$\begin{aligned}
& \alpha_i \left(\frac{\lambda - 1}{\lambda}\right)(\gamma + \rho) = \rho + \tau_i \\
\Rightarrow \quad & \tau_i^* = \alpha_i \left(\frac{\lambda - 1}{\lambda}\right)(\gamma + \rho) - \rho
\end{aligned}$$

Note that we assume α_i is bounded from below by the appropriate factor to ensure τ_i is positive for all i.

(c) Substituting, we can see that production is

$$y_i = \alpha_i q_i P$$

Plugging this into the final good aggregator, we find

$$y = \Lambda Q P$$

where

$$\log(\Lambda) = \int_0^1 \alpha_i \log(\alpha_i)\ di \quad \text{and} \quad \log(Q) = \int_0^1 \alpha_i \log(q_i)\ di$$

Thus growth will be driven by growth in Q. This works out to

$$g = \int_0^1 \tau_i \alpha_i \log(\lambda)\ di = \log(\lambda)\mathbb{E}[\alpha_i \tau_i]$$

So we finally find

$$g^* = \log(\lambda)\left[\sigma\left(\frac{\lambda-1}{\lambda}\right)(\gamma+\rho) - \rho\right]$$

2 Patents and Innovation

(a) Here we need to differentiate between those product lines with unexpired patents (monopolists) and those with expired patents (competitive). The case of the monopolist will be similar to what we've seen thus far. The optimization problem of the final good aggregator yields the inverse demand function

$$p_i = \frac{\partial y}{\partial y_i} = \frac{y}{y_i}$$

In the case a monopolist, limit pricing dictates that we set price to the marginal cost of the competitor

$$p_i^M = \frac{w\lambda}{q_i} \quad \Rightarrow \quad y_i^M = \frac{q_i y}{\lambda w}$$

Thus labor utilization is

$$\ell_i^M = \lambda^{-1}\frac{y}{w} \quad \Rightarrow \quad \pi_i^M = (1-\lambda^{-1})y$$

In the case of a competitive product line, the outcome will look like a case where $\lambda = 1$ and firms will price at marginal cost. Thus we find

$$p_i^C = \frac{q}{q_i} \quad \Rightarrow \quad y_i^C = \frac{q_i y}{w} \quad \Rightarrow \quad \ell_i^C = \frac{y}{w} \quad \Rightarrow \quad \pi_i^C = 0$$

Given a fraction of monopolistic product lines μ, labor market clearing then implies

$$\begin{aligned} P &= \int_0^1 \ell_i di = \frac{y}{w}\left[\lambda^{-1}\mu + (1-\mu)\right] \\ \Rightarrow \quad w &= \frac{y}{P}\left[1 - \mu(1-\lambda^{-1})\right] \end{aligned}$$

(b) We can write a flow equation for μ according to

$$\dot{\mu} = \tau(1-\mu) - b\mu$$

In steady state, we then find

$$\dot{\mu} = 0 \quad \Rightarrow \quad \mu^* = \frac{\tau}{b+\tau}$$

(c) When an innovator is successful, they become a monopolist, at least initially. So we should use monopoly profit levels for the value of an innovation. Thus we get

$$rv_i - \dot{v}_i = \pi_i^M - (b+\tau)v_i \quad \Rightarrow \quad v_i = \frac{\pi_i^M}{r - g_v + b + \tau} = \frac{\pi_i^M}{\rho + b + \tau}$$

Now we can use this in the free entry condition

$$\gamma v_i = w \quad \Rightarrow \quad \frac{\gamma \pi_i^M}{\rho + b + \tau} = w$$

Substituting in from previous parts, we find

$$\begin{aligned} \frac{\gamma(1-\lambda^{-1})y}{\rho+b+\tau} &= \frac{y}{P}\left[1-\mu(1-\lambda^{-1})\right] \\ \frac{\lambda-1}{\rho+b+\tau} &= \frac{1}{\gamma-\tau}\left[\lambda - \frac{\tau}{b+\tau}(\lambda-1)\right] \end{aligned}$$

This is actually a quadratic in τ, though the exact solution is unlikely to be illuminating. We can, however, see that the left-hand side will be decreasing in b and the right-hand side will be increasing in b, so their intersection should be shifted lower as b increases. Thus a higher patent expiry rate b will lead to lower values for τ and hence for g.

Furthermore, as $b \to \infty$, the left-hand side goes to zero, while the right-hand side converges to a constant, so the equilibrium τ^* must go to zero as well.

(d) Getting anything analytically tractable seems hopeless here. Nonethless, if we are in a case where there is overinvestment in innovation in equilibrium ($\tau^* > \hat{\tau}$), we could employ patents of finite length ($b > 0$) to reduce τ^* to more efficient levels. From part (c), we know there must be some level of b that yields exactly $\tau^* = \hat{\tau}$.

However, this expiry rate (call it b^*) will not yield the same level of welfare as that achieved by the social planner, as we still experience some deadweight monopoly loss in equilibrium. Recall that the social planner sets labor equally across all product lines, while in equilibrium there will be some difference between monopolistic and competitive product lines. Thus the truly optimal choice for b would likely be somewhere a bit above b^*.

In the case where there is too little innovation in equilibrium even with infinite length patents, there is not much more we can do. In that case, one might consider a direct subsidy to research.