PS1: Solutions

1 Future Malthus

(a) This graph will start at n_2 for $y = 0$ then decrease linearly until it hits n_1 at $y = \frac{n_2 - n_1}{\theta}$ and will stay there for any higher level of y.

This is in line with what we see for modern era where income levels are generally higher. The source can be either birth rates or death rates. It would be surprising if overall death rates increased with income, so it is more likely from declining birth rates. Here, possible mechanisms include a reduced demand for within-family agricultural labor, changes in religious beliefs, an increased opportunity cost of time spent on child rearing, changes in the empowerment of women, and many more.

(b) From the aggregate production function we can find the per capita income level

$$y = \frac{Y}{L} = zK^{\alpha}L^{-\alpha}$$

Using the rules of growth rates and recalling that $g_K = 0$, this means

$$g_y = g_z - \alpha g_L$$

Thus we will expect to see positive income growth when

$$g_L < \frac{g_z}{\alpha}$$

We are left with three cases, depending on parameters

- Case 1: $\frac{g_z}{\alpha} > n_2$ — in this case, the above inequality will always hold, so we will have growth in y no matter what. In the long run, we will end up with $y \to \infty$ and $g_y = g_z - \alpha n_1 > 0$.

- Case 2: $\frac{g_z}{\alpha} < n_1$ — in this case, the above inequality never holds, so we will always have negative growth in y. In the long run, we will end up with $y \to 0$ and $g_y = g_z - \alpha n_2 < 0$.

- Case 3: $n_2 > \frac{g_z}{\alpha} > n_1$ — in this we will end up with either the Case 1 or Case 2 outcome, but it will depend on our initial value for y.

Focusing on the more interesting Case 3, the critical value for y can be found by seeing where $g_y = 0$, meaning

$$0 = g_y = g_z - \alpha g_L = g_z - \alpha(n_2 - \theta y)$$
$$\Rightarrow \quad y^* = \frac{1}{\theta}\left(n_2 - \frac{g_z}{\alpha}\right)$$

Recall that the initial value for y is determined from the per capita income equation and will depend on the initial values for both technology z and population L.

(c) In the bad equilibrium where $y \to 0$, this will look roughly like a decreasing exponential over time. In the good equilibrium where $y \to \infty$, this will look like an increasing exponential over time. Fancy that! In the short run, things won't be precisely exponential, but it would likely be a pretty subtle difference. The exact long run growth rates are those given above.

(d) In a setting with competitive markets, we would expect the returns on capital (the interest rate r) and labor (the wage rate w) to be close to their respective marginal products. So here that means

$$r = \frac{\partial Y}{\partial K} = \alpha z K^{\alpha-1} L^{1-\alpha}$$
$$w = \frac{\partial Y}{\partial L} = (1-\alpha) z K^{\alpha} L^{-\alpha}$$

There are many ways to answer this question, but one approach is to think about this as a labor choice. You can either spend your time exploring/conquering to find new land or working to produce goods. In the exploration case, let's say you find one unit of new land which has marginal value r. In the production case, you earn your marginal product w. So which of these you choose will depend on which is larger, r or w. Thus, we'll look at the ratio of the two

$$\frac{r}{w} = \frac{\alpha}{1-\alpha}\frac{L}{K}$$

Since α and K are constant, and L is growing exponentially, this ratio will growing exponentially as well. Thus the exploration option will look more and more profitable relative to production labor over time. For this reason, the constant land assumption might eventually break down.

2 Endogenous Growth

(a) This production function is consistent with a setting where each person is just continually thinking up new ideas at random. Thus the larger the population is, the more new ideas there are. This is a good first approximation.

Nonetheless, there may be reasons to question it. For instance, it may be the with enough people you start getting duplicate ideas, which would induce decreasing returns to scale, rather than the linear relationship we have here.

Alternatively, it could be that ideas are in fact costly to generate and require effort. In this case we should think about what people's incentive is to come up with new ideas. We do so in part (d) and find that this may still be reasonable!

(b) To find the growth rate of z, we simply divide to find

$$g_z = \frac{\dot{z}}{z} = \frac{\eta L}{z}$$

Thus in the long-run, the ratio L to z should be constant. To achieve this, we thus require that $g_z = g_L$, which we can see from the quotient rule for growth rates.

(c) First, we'll do a proof by contradiction to rule out the bad equilibrium, then we'll characterize the dynamics a bit more. Suppose that we are in the bad equilibrium, where $y \to 0$. Then it must be that the long-run growth rate of y is negative. Additionally, we will converge to $g_L = n_2$. Using the result from part (b), this means

$$g_y = g_z - \alpha g_L = (1-\alpha) g_L = (1-\alpha) n_2 > 0$$

A contradiction, so we cannot actually be in this equilibrium. That's good enough for a solid answer to the question.

But let's try to characterize the evolution of things in the short-run. We'll define a new normalized technology parameter $q = z/L$. The growth rate of this quantity will be

$$\frac{\dot{q}}{q} = g_z - g_L = \frac{\eta}{q} - g_L$$

$$\Rightarrow \quad \dot{q} = \eta - g_L \cdot q$$

Meanwhile, we can write a law of motion for y with

$$\frac{\dot{y}}{y} = g_z - \alpha g_L = \frac{\eta}{q} - \alpha g_L$$

Combining these, we then have a system of two differential equations in (y, q)

$$\frac{\dot{y}}{y} = \frac{\eta}{q} - \alpha g_L(y)$$

$$\frac{\dot{q}}{q} = \frac{\eta}{q} - g_L(y)$$

where I'm writing $g_L(y)$ to clarify its direct dependence on y. In the long-run, we expect q to converge to some constant with $\dot{q} = 0$. Once that occurs we recover our previous equation

$$\frac{\dot{y}}{y} = (1 - \alpha) g_L(y) > 0$$

Meaning $g_z = g_L = n_1$, $g_y = (1 - \alpha)n_1$, and $q = \frac{\eta}{n_1}$.

(d) Here we can consider the marginal product of additional technology

$$\frac{\partial Y}{\partial z} = K^\alpha L^{1-\alpha}$$

As before, we can consider the choice of doing research and generating one new idea with marginal product $\frac{\partial Y}{\partial z}$ versus the option of doing production and earning wage w. So we'll look at the ratio

$$\frac{\frac{\partial Y}{\partial z}}{w} = (1 - \alpha)\frac{L}{z}$$

Thus this will actually converge to a constant in the long-run, meaning it is reasonable to think people will choose one or the other depending on their personal preferences and ability. Specifically, we found in part (c) that we have

$$q = \frac{z}{L} \to \frac{\eta}{n_1}$$

and so the ratio above will converge to $(1 - \alpha)\frac{n_1}{\eta}$.

3 Dynamic Stability

(a) See lecture 2.

(b) Here, the growth rate of capital is

$$\frac{\dot{K}}{K} = sA - \delta$$

Thus capital grows at a constant rate regardless of the initial value, meaning

$$K(t) = K_0 \exp((sA - \delta)t)$$

(c) For any production function, the dynamics of k are given by

$$\dot{k} = sf(k) - \delta k$$

Here we have

$$f(k) = \left[\alpha k^{1-\rho} + (1-\alpha)\right]^{\frac{1}{1-\rho}}$$

and hence

$$\dot{k} = s\left[\alpha k^{1-\rho} + (1-\alpha)\right]^{\frac{1}{1-\rho}} - \delta k \equiv g(k)$$

First note that $g(0) = s(1-\alpha)^{\frac{1}{1-\rho}} > 0$, meaning $k = 0$ is not stable. There is a solution to $g(k) = 0$ at

$$k^* = \left[\frac{1-\alpha}{\left(\frac{\delta}{s}\right)^{1-\rho} - \alpha}\right]^{\frac{1}{1-\rho}}$$

This is positive and well defined so long as condition A holds:

$$\alpha\left(\frac{s}{\delta}\right)^{1-\rho} < 1 \qquad \text{(Condition A)}$$

Notice that this places a lower bound on s for $\rho < 1$, no bound s for $\rho = 1$, and an upper bound on s for $\rho > 1$. It turns out the condition for stability is also condition A. Note that

$$g'(k) = \alpha s\left[\alpha + (1-\alpha)k^{\rho-1}\right]^{\frac{\rho}{1-\rho}} - \delta$$

Evaluating this at the steady state yields

$$g'(k^*) = \delta\left[\alpha\left(\frac{s}{\delta}\right)^{1-\rho} - 1\right]$$

which yields the stability condition $g'(k^*) < 0$ under condition A.

Solving for the transition path in closed form seems difficult. If we try for an analogous substitution as for the Cobb-Douglas case, namely

$$z = \frac{k}{f(k)} = \left[\alpha + (1-\alpha)k^{\rho-1}\right]^{\frac{1}{\rho-1}}$$

we get the equation

$$\dot{z} = (1 - \alpha z^{1-\rho})(s - \delta z)$$

which is a linear differential equation for $\rho = 1$ but otherwise rather intractable.

4 Balanced Growth Paths

(a) First we can take a total derivative of the production function to get a growth rate rule

$$g_Y = \alpha \left(\frac{K}{Y}\right)^{1-\rho} g_K + (1-\alpha)\left(\frac{AL}{Y}\right)^{1-\rho} (g+n)$$

and from the capital accumulation equation

$$g_K = \frac{sY}{K} - \delta$$

Thus so long as we have positive shares of capital and labor, we need

$$g_Y = g_K = g + n$$

So we normalize all variables by AL. Now it is simply a matter of converting equations

$$\begin{aligned} y &= \left[\alpha k^{1-\rho} + (1-\alpha)\right]^{\frac{1}{1-\rho}} \\ \dot{k} &= sy - (\delta + g + n)k \end{aligned}$$

(b) Here our growth rate rule is

$$g_Y = \alpha g_K + \beta(g+n)$$

which in conjunction with the law of motion for capital yielding $g_Y = g_K$ yields

$$g_Y = g_K = \left(\frac{\beta}{1-\alpha}\right)(g+n) \equiv \bar{g}_K(g+n)$$

So we normalize all variables by $(AL)^{\frac{\beta}{1-\alpha}}$. Plugging this into the production function we get

$$y(AL)^{\frac{\beta}{1-\alpha}} = k^\alpha (AL)^{\frac{\alpha\beta}{1-\alpha}} (AL)^\beta \quad \Rightarrow \quad y = k^\alpha$$

and

$$\dot{k} = sy - (\delta + \bar{g}_K(g+n))k$$

(c) Here our production function yields the familiar

$$g_Y = \alpha g_K + (1-\alpha)(g+n)$$

Looking at the capital accumulation equation, we get

$$g_K = \frac{I^\gamma}{K^{1-\beta}} = \frac{(sY)^\gamma}{K^{1-\beta}}$$

which requires that

$$\gamma g_Y = (1-\beta)g_K \quad \Rightarrow \quad g_Y = \left(\frac{1-\beta}{\gamma}\right) g_K$$

The condition that $\beta + \gamma > 1$ thus implies $g_K > g_Y$. Solving these two growth equations jointly yields

$$g_I = g_Y = \frac{(1-\alpha)(g+n)}{\frac{1-\beta}{\gamma} - \alpha} = \frac{\gamma(1-\alpha)(g+n)}{1-\beta-\alpha\gamma} \equiv \bar{g}_Y(g+n)$$

and

$$g_K = \frac{(1-\beta)(1-\alpha)(g+n)}{1-\beta-\alpha\gamma} \equiv \bar{g}_K(g+n)$$

So in fact we require

$$\gamma > 1-\beta > \alpha\gamma$$

and we can see that

$$g_K > g_Y > g+n$$

And we normalize our variables by their respective growth rates. Finally reformulating our equations, we get

$$y(AL)^{\bar{g}_Y} = k^\alpha (AL)^{\alpha \bar{g}_Y} (AL)^{1-\alpha} \quad \Rightarrow \quad y = k^\alpha$$

and

$$\begin{aligned}
\frac{\dot{k}}{k} &= \frac{I^\gamma}{K^{1-\beta}} - \bar{g}_K(g+n) \\
&= \frac{i^\gamma (AL)^{\gamma \bar{g}_Y}}{k^{1-\beta}(AL)^{(1-\beta)\bar{g}_K}} - \bar{g}_K(g+n) \\
&= \frac{i^\gamma}{k^{1-\beta}} - \bar{g}_K(g+n) \\
\Rightarrow \dot{k} &= k^\beta i^\gamma - \bar{g}_K(g+n)k
\end{aligned}$$