Solutions: Midterm Review

1 Workers and Capitalists

(a) First let's deal with the capitalists. Their asset evolution equation divides cleanly to

$$\dot{B} = rB - C_k \quad \Rightarrow \quad \dot{a} = rb - c_k$$

Turning to the workers we get

$$C_w = w(1-\beta)L \quad \Rightarrow \quad c_w = w$$

Now for factor prices. Here we have (with $r = R - \delta$)

$$\begin{aligned} R &= \alpha K^{\alpha-1}[(1-\beta)L]^{1-\alpha} = \alpha k^{\alpha-1} \\ w &= (1-\alpha)K^{\alpha}[(1-\beta)L]^{-\alpha} = (1-\alpha)k^{\alpha} \end{aligned}$$

Here the factor shares satisfy

$$RK + [(1-\beta)L]w = K^{\alpha}[(1-\beta)L]^{1-\alpha}$$

and production satisfies $y = k^{\alpha}$, where $y = Y/((1-\beta)L)$.

Utility is logarithmic, so we can use standard arguments to show

$$\frac{\dot{c}_k}{c_k} = r - \rho$$

Finally, we equate assets and capital. Here $B = K$ implies

$$\beta b = (1-\beta)k$$

Using the capitalists asset evolution, we get

$$
\begin{aligned}
\dot{k} &= rk - \left(\frac{\beta}{1-\beta}\right) c_k \\
&= \alpha k^{\alpha} - \delta k - \left(\frac{\beta}{1-\beta}\right) c_k
\end{aligned}
$$

Notice that we can transform this into

$$
\begin{aligned}
\dot{k} &= k^{\alpha} - \delta k - \left(\frac{\beta}{1-\beta}\right) c_k - (1-\alpha)k^{\alpha} \\
&= y - \delta k - \left(\frac{\beta}{1-\beta}\right) c_k - c_w
\end{aligned}
$$

(b) In steady state we have $r = \rho$ so that

$$
k^* = \left(\frac{\alpha}{\rho + \delta}\right)^{\frac{1}{1-\alpha}}
$$

The same as usual. As for capitalist consumption

$$
\begin{aligned}
c_k &= \left(\frac{1-\beta}{\beta}\right) (\alpha k^{\alpha} - \delta k) \\
&= \rho \left(\frac{1-\beta}{\beta}\right) \left(\frac{\alpha}{\rho + \delta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

And the workers get

$$
\begin{aligned}
c_w &= (1-\alpha)k^{\alpha} \\
&= (1-\alpha) \left(\frac{\alpha}{\rho + \delta}\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

If we look at the total consumption share, we get

$$
\begin{aligned}
s_c = \frac{\beta L c_k + (1-\beta) L c_w}{(1-\beta) L y} &= \frac{\rho k + (1-\alpha) k^\alpha}{k^\alpha} \\
&= 1 - \frac{\alpha\delta}{\rho+\delta}
\end{aligned}
$$

This means the savings rate is the same as in the basic Ramsey model without population or technological growth, namely

$$
s_i = \frac{\alpha\delta}{\rho+\delta}
$$

If we look at the fraction of income going to capitalists, we get

$$
\begin{aligned}
s_k = \frac{\beta L c_k}{\beta L c_k + (1-\beta) L c_w} &= \frac{\rho k}{\rho k + (1-\alpha) k^\alpha} \\
&= \frac{\alpha\rho}{\rho + (1-\alpha)\delta}
\end{aligned}
$$

and thus that going to workers is

$$
s_w = \frac{(1-\alpha)(\rho+\delta)}{\alpha\rho + (1-\alpha)\delta}
$$

So aggregate shares don't change with β. What does change is output per capita (it goes down with β)

$$
\frac{Y}{L} = (1-\beta) y = (1-\beta)\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}}
$$

and the consumption of individual capitalists c_k (it goes down with β).

(c) We can characterize the dynamics with k and c_k. These equations are

$$
\begin{aligned}
\dot{k} &= \alpha k^\alpha - \delta k - \left(\frac{\beta}{1-\beta}\right) c_k \\
\dot{c}_k &= c_k[\alpha k^{\alpha-1} - (\rho+\delta)]
\end{aligned}
$$

Taking a first order Taylor expansion around steady state values, we arrive at

$$J^* = \begin{bmatrix} \alpha\rho - (1-\alpha)\delta & -\frac{\beta}{1-\beta} \\ -(1-\alpha)\rho(\rho+\delta)\left(\frac{\beta}{1-\beta}\right) & 0 \end{bmatrix}$$

The determinant of this matrix is

$$\det(J^*) = -(1-\alpha)\rho(\rho+\delta)\left(\frac{\beta}{1-\beta}\right)^2 < 0$$

Since the determinant is the product of the eigenvalues, it must be that one is negative and one is positive, meaning there are saddle path dynamics so long as $\beta > 0$.

(d) Assuming $s_k > \beta$, we have a rich fraction of society of size β that gets a fraction s_k of consumption, and a poor fraction of size $1-\beta$ that gets a fraction $s_w = 1 - s_k$ of consumption.

Thus the Lorenz is a piecewise linear function passing through $(0,0)$, $(1-\beta, s_w)$, and $(1,1)$. The area under this curve is thus

$$\int L = \frac{1}{2}(s_w + \beta) = \frac{1}{2}(1 - (s_k + \beta))$$

The Gini coefficient is then

$$G = \frac{\frac{1}{2} - \int L}{\frac{1}{2}} = \beta + s_k$$

We saw the share going to capitalists is not a function of β, so then the effect of β is positive and linear, as above. For reasonable parameters, we get $s_k = 20\%$. Supposing $\beta = 10\%$, then the Gini coefficient would be 30%, which is a bit lower than the US value of 40%.

2 Labor and Leisure

(a) The production function implies

$$
\begin{aligned}
g_Y &= \alpha g_K + (1-\alpha)(g_A + g_h + g_L) \\
&= \alpha g_K + (1-\alpha)(g_h + g + n)
\end{aligned}
$$

In aggregate we still have the same equation for the evolution of capital

$$
\dot{K} = I - \delta K \quad \Rightarrow \quad g_K = \frac{I}{K} - \delta
$$

which assuming constancy of g_K implies

$$
g_I = g_K
$$

Finally, assuming interior shares of consumption and savings gives us $g_Y = g_I = g_C$, which using the above implies

$$
g_K = g + n + g_h
$$

We will exclude the case where h has a negative growth rate and converges to zero (we can check this later). In this case, since it is bounded, its growth rate must be zero as well. Thus we get the usual expression

$$
g_Y = g_K = g_C = g_I = g + n
$$

Thus $g_c = g$.

(b) Given the above answer, it is natural to normalize by something that grows at rate $g + n$, namely, AL. Reformulating the production function using normalized variables yields

$$
y = k^{\alpha} h^{1-\alpha}
$$

In terms of assets held by consumers, letting $\tilde{w} \equiv w/A$ and $\tilde{c} \equiv c/A$, we get (since A already taken, let B be total assets and $b = B/(AL)$ for now)

$$\dot{B} = rB + whL - cL$$
$$\Rightarrow \dot{b} = (r - (g+n))b + \tilde{w}h - \tilde{c}$$

It is useful to define an effective interest rate $\hat{r} \equiv r - (g+n)$, in which case

$$\dot{b} = \hat{r}b + \tilde{w}h - \tilde{c}$$

After substituting in $\tilde{c}$, our effective discount rate is now $\hat{\rho} = \rho - n - \eta(1-\theta)g$. Notice we are ignoring the -1 in the numerator of utility, but this will have no effect, since it is just a constant. The Hamiltonian of the commune is then

$$H = u(\tilde{c}, 1-h) + \mu(\hat{r}b + \tilde{w}h - \tilde{c})$$

The FOCs for the control variables are

$$H_c = 0 \quad \Rightarrow \quad u_c = \mu$$
$$H_h = 0 \quad \Rightarrow \quad u_\ell = \tilde{w}\mu$$

which in particular implies $\tilde{w}u_c = u_h$. The utility derivatives are

$$u_c = \eta \left(\frac{\ell}{\tilde{c}}\right)^{1-\eta} \left[\tilde{c}^\eta \ell^{1-\eta}\right]^{-\theta}$$
$$u_\ell = (1-\eta) \left(\frac{\tilde{c}}{\ell}\right)^{\eta} \left[\tilde{c}^\eta \ell^{1-\eta}\right]^{-\theta}$$

From the above, this then implies

$$\eta\tilde{w}\ell = (1-\eta)\tilde{c}$$

From this we can see how the Cobb-Douglas inner form factors in. Consider the following rearrangement of the asset equation

$$\tilde{c} + \tilde{w}\ell = \hat{r}b + \tilde{w} - \dot{b}$$

The left side is spending on consumption good (price 1) and on leisure good (price w). The right side is the maximum possible income ($h = 1$) of the agent *conditional* on investment decisions ($\dot{a}$). Substituting using the labor FOC, we find

$$\tilde{c} = \eta(\hat{r}b + \tilde{w} - \dot{b})$$
$$\tilde{w}\ell = (1 - \eta)(\hat{r}b + \tilde{w} - \dot{b})$$

So η and $1 - \eta$ are the spending shares for each of these "goods", as we usually find with Cobb-Douglas utility.

The second Hamiltonian optimality condition is

$$\hat{\rho}\mu - \dot{\mu} = H_b = \hat{r}\mu \quad \Rightarrow \quad -\frac{\dot{\mu}}{\mu} = r - \rho - (\theta + (1 - \eta)(1 - \theta))g$$

As usual, we can use the consumption FOC to find

$$-\frac{\tilde{c}u_{cc}}{u_c}\frac{\dot{\tilde{c}}}{\tilde{c}} = -\frac{\dot{\mu}}{\mu} = r - \rho$$

which implies

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{r - \rho - (\theta + (1 - \eta)(1 - \theta))g}{\theta}$$

Notice that when $\theta = 1$, η does not enter into the Euler equation, and when $\eta = 1$, we get what we found in lecture.

(c) First, we need to know the factor prices. If we think of the firm as choosing total hours of work $H = hL$, their objective is

$$\Pi = K^{\alpha}(AH)^{1-\alpha} - wH - RK$$

This yields factor prices

$$
\begin{aligned}
R = \alpha K^{\alpha-1}(AH)^{1-\alpha} &= \alpha \left(\frac{h}{k}\right)^{1-\alpha} \\
\Rightarrow r = \alpha \left(\frac{h}{k}\right)^{1-\alpha} - \delta
\end{aligned}
$$

and

$$
\begin{aligned}
w &= A(1-\alpha)K^{\alpha}(AH)^{-\alpha} \\
\Rightarrow \tilde{w} &= (1-\alpha)\left(\frac{k}{h}\right)^{\alpha}
\end{aligned}
$$

From here we can see that

$$
Rk + \tilde{w}h = k^{\alpha}h^{1-\alpha}
$$

Equating capital and assets we get

$$
\dot{k} = k^{\alpha}h^{1-\alpha} - (\delta + g + n)k - \tilde{c}
$$

In steady state, this implies

$$
\tilde{c} = k^{\alpha}h^{1-\alpha} - (\delta + g + n)k
$$

The other two relevant equations are the hours worked FOC and the Euler equation. Letting $\hat{\theta} \equiv \theta + (1-\eta)(1-\theta)$, we get

$$
\eta\tilde{w}(1-h) = (1-\eta)\tilde{c}R = \delta + \rho + \hat{\theta}g
$$

Recall that factor shares satisfy $Rk = \alpha y$ and $\tilde{w} = (1-\alpha)y$, thus we can write the system of equations

$$
\tilde{c} = y - (\delta + g + n)k\eta(1-\alpha)y\left(\frac{1-h}{h}\right) = (1-\eta)\tilde{c}\alpha\frac{y}{k} = \delta + \rho + \hat{\theta}g
$$

From here we find

$$\frac{k}{y} = \frac{\alpha}{\delta + \rho + \hat{\theta} g}$$

and hence

$$\frac{\tilde{c}}{y} = 1 - \alpha \left(\frac{\delta + g + n}{\delta + \rho + \hat{\theta} g} \right) \equiv 1 - s$$

where s refers to the savings rate. And finally

$$\frac{1-h}{h} = \frac{1}{1-\alpha} \left(\frac{1-\eta}{\eta} \right) (1-s)$$

meaning

$$h = \frac{\eta(1-\alpha)}{\eta(1-\alpha) + (1-\eta)(1-s)}$$

From here we can get

$$k = \left(\frac{\alpha}{\delta + \rho + \hat{\theta}} \right)^{\frac{1}{1-\alpha}} h$$

For the stability part, we would find the Jacobian of (k, c) system around steady state. To deal with h, we would have to use the labor FOC to account for the dependence of h on c. Then we would find the eigenvalues of the Jacobian and see if they have opposite signs.

(d) Let $v(\ell) = -\psi(1-\ell)^{\gamma}$. In normalized terms, our period utility is

$$u(\tilde{c}, \ell) = \log(A\tilde{c} + v(\ell)) = \log(A) + \log(\tilde{c} + v(\ell)/A)$$

The marginal utilities are then

$$u_c = \frac{A}{A\tilde{c} + v(\ell)}$$
$$u_\ell = \frac{v'(\ell)}{A\tilde{c} + v(\ell)}$$

The optimality condition thus yields

$$w = A\tilde{w} = v'(\ell)$$

Since the marginal utility of leisure is falling, workers will eventually choose $h = 1$. Thus our optimality condition will really be an inequality at the boundary $\ell = 0$

$$A\tilde{w} > v'(0) = \gamma\psi$$

Since A is growing without bound, then will eventually hold and we will have $h = 1$. The rest of the work above will still go through using $\theta = 1$ (where CRRA goes to log) and $h = 1$, so that

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = r - \rho - g$$

and

$$\dot{k} = k^\alpha - (\delta + g + n)k - \tilde{c}$$

as we found in lecture.

Note: Things are actually a little more interesting when $g = 0$ (meaning $A(t) = 1$) since you get

$$w = v'(\ell)$$

Thus labor and leisure can be found using only the wage and do *not* depend on c, which makes things easier. This would be a good practice exercise!

3 Rise of the Machines

(a) The growth rate of capital is

$$\dot{k} = s_k y - (\delta + n)k$$

The growth rate of robots is

$$\dot{m} = s_m y - (\delta + n)m$$

The production function is

$$y = k^{\alpha}(1+m)^{1-\alpha}$$

(b) The growth rate of capital is

$$g_k = \frac{\dot{k}}{k} = s_k \left(\frac{m}{k}\right)^{1-\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - (\delta + n)$$

The growth rate of robots is

$$g_m = \frac{\dot{m}}{m} = s_m \left(\frac{k}{m}\right)^{\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - (\delta + n)$$

The growth rate of output is

$$\begin{aligned} \dot{y} &= \alpha \left(\frac{y}{k}\right) \dot{k} + (1-\alpha) \left(\frac{y}{1+m}\right) \dot{m} \\ \Rightarrow \quad \frac{\dot{y}}{y} &= \alpha \frac{\dot{k}}{k} + (1-\alpha) \left(\frac{m}{1+m}\right) \frac{\dot{m}}{m} \\ \Rightarrow \quad g_y &= \alpha g_k + (1-\alpha) \left(\frac{m}{1+m}\right) g_m \end{aligned}$$

(c) In steady state, k and m should have the same growth rate ($g_k = g_m$)

$$s_k \left(\frac{m}{k}\right)^{1-\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - (\delta + n) = s_m \left(\frac{k}{m}\right)^{\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - (\delta + n)$$

which implies

$$\frac{s_k}{s_m} = \frac{k}{m}$$

and hence

$$g_k = g_m = s_k^{\alpha} s_m^{1-\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - (\delta + n)$$

There are two cases. The first is that both k and m grow without bound at rate

$$g = s_k^{\alpha} s_m^{1-\alpha} - (\delta + n)$$

However, when this figure is negative, we get no growth ($g_k = g_m = 0$) in the limit, and

$$m = \frac{1}{\left(\frac{\delta+n}{s_k^{\alpha} s_m^{1-\alpha}}\right)^{\frac{1}{1-\alpha}} - 1}$$

and

$$k = \frac{1}{\left(\frac{\delta+n}{s_k}\right)^{\frac{1}{1-\alpha}} - \frac{s_m}{s_k}}$$

(d) The marginal product of labor is

$$\begin{aligned} \frac{\partial Y}{\partial L} &= (1-\alpha)\left(\frac{K}{L+M}\right)^{\alpha} \\ &= (1-\alpha)\left(\frac{k}{1+m}\right)^{\alpha} \\ &= (1-\alpha)\left(\frac{k}{m}\right)^{\alpha}\left(\frac{m}{1+m}\right)^{\alpha} \end{aligned}$$

So in the continual growth case, we get

$$\frac{\partial Y}{\partial L} = (1-\alpha)\left(\frac{s_k}{s_m}\right)^{\alpha}$$

In the no growth case, we get

$$\begin{aligned}\frac{\partial Y}{\partial L} &= (1-\alpha)\left(\frac{s_k}{s_m}\right)^{\alpha}\left(\frac{s_k^{\alpha}s_m^{1-\alpha}}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} \\ &= (1-\alpha)\left(\frac{s_k}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}\end{aligned}$$