Problem Set 1: Malthus

Please submit your solutions to Canvas by 11:59pm on Friday, February 7th. You are encouraged to work together on these assignments, but submit only your own work.

1 Malthus + Ecology

Imagine a society in which production is undertaken using both labor and some replenishable ecological resource. One prime example would be some kind of hunter-gatherer society. Let the ecological resource be doneted by $E > 0$, and suppose that it naturally replenishes at rate γ and is depleted by humans at rate δ, so that

$$\dot{E} = \gamma E - \delta L$$

The production function is then given according to

$$Y = zE^{\alpha}L^{1-\alpha}$$

Assume that the level of technology (z) is not growing. The population growth rate is classic Malthus with

$$g_L = \theta(y - \bar{y})$$

where $g_L = \dot{L}/L$ and $y = Y/L$.

(a) It makes sense to think about this system in terms of $e = E/L$. Derive expressions for g_E, g_L, and y purely in terms of e and model parameters.

(b) Plot g_E and g_L on the same graph as functions of e. The intersection points represent where $g_e = 0$. Describe the possible stable and unstable steady states of this model. What is the long-run behavior of population L and the standard of living y in these cases? [**Note**: there aren't closed form expressions for the steady state values of e, just depict these graphically.]

(c) Derive sufficient conditions for there to be a stable steady state with $e > 0$. It may be helpful to find the points where the growth curves cross zero, that is those satisfying $g_E(e_E^*) = 0$ and $g_L(e_L^*) = 0$.

(d) What is the effect of a one time increase in technology z? Specifically, how do the long-run values for e, g_L, and y change? Does the answer depend on the size of the change in z?

2 Malthus + Migration

Consider the case of a Malthusian world where there two distinct regions $i \in \{1, 2\}$. Each region has a fixed amount of land K_i and population levels L_i that change in response to various forces. They also have productivity levels z_i that enter into a Cobb-Douglas production function

$$Y_i = z_i K_i^{\alpha} L_i^{1-\alpha}$$

As usual, the rate of population growth is a function of the standard of living $y_i = Y_i/L_i$. Denote this relationship by $\dot{L}_i/L_i = h(y_i)$, where h is a strictly increasing function with $h(\bar{y}) = 0$, which encompasses the usual linear case.

(a) Find an equation relating the standard of living y_i to the population L_i. In simple terms, what is the meaning of this equation?

(b) Suppose that the two regions are completely isolated. Plot the standard of living y_i as a function of population L_i and use this graph to solve for the steady state population. Draw arrows on the graph depicting convergence dynamics.

(c) Now consider the case where productivity z_i is actually endogenous and is increasing in the population density. Let this relationship take the form $z_i = \phi(L_i/K_i)$, where ϕ is a strictly increasing function with $\phi(0) = 0$ and $\phi(\infty) = \bar{z}$. Now draw the same type of graph as in part (b) and describe the steady state(s) of the model. Are there different answers depending on functional or parametric assumptions?

(d) Next consider the case of inter-regional migration. Suppose the total population is fixed with $L = L_1 + L_2$, but people can move between regions. People will tend to move towards the region with a higher standard of living y_i over time. Draw the same type of graph as in parts (b) and (c) but with L_2 mirrored (like in an Edgeworth box) and use this to find the possible steady states.

3 Kremer Loose Ends [OPTIONAL]

Now consider an extension of the above problem in which we partially endogenize the rate of technological growth. In particular, let the increase in z be proportional to the population size, so that we have

$$\dot{z} = \eta L$$

for some $\eta > 0$. Call this the technology production function.

(a) What are some reasons why we might expect a technology production function of the form seen above? Can you come up with a rough estimate for η in the modern context?

(b) For a given L and z, what is the growth rate of technology g_z? In the long run, what does this imply about the ratio of L and z?

(c) Prove that long run technological growth is inevitable in this model, meaning immiseration is not possible. Find an expression for the long-run growth rate of technology z and the standard of living y?

(d) Similar to the previous question, find the marginal value of an incremental improvement in technology z and consider its implications for the given technology production function.