Problem Set 1

Please submit your solutions to Canvas by 10:30am on Tuesday, February 9th. You are encouraged to work together on these assignments, but submit only your own work.

1 Future Malthus

Consider a Malthusian model where we lie of the right side of the demographic transition. Now, as we see in modern economies, population growth $\frac{\dot{L}}{L}$ is inversely related to the standard living $\frac{Y}{L}$

$$\frac{\dot{L}}{L} = \max\left\{n_1,\ n_2 - \theta\frac{Y}{L}\right\}$$

where $n_2 > n_1 > 0$ and $\theta > 0$. As before, we have a Cobb-Douglas production function that uses land K and labor L with technology level z

$$Y = zK^{\alpha}L^{1-\alpha}$$

(a) Plot population growth function as we did in class. What are some reasonable explanations for such a relationship?

(b) Supposing that technology grows as rate $\frac{\dot{z}}{z} = g_z$, under what conditions can we expect long run growth in the standard of living $\frac{Y}{L}$? Does the answer depend on the initial value for the standard of living?

(c) Plot some example paths of $\frac{\dot{L}}{L}$ and $\frac{Y}{L}$. Do this both for a case where we see continual growth and a case where we see stagnation.

(d) Find expressions for the marginal products of land and labor. Assuming competitive markets, meaning these are the rent r and wage w, argue that this is an unrealistic model of modern growth.

2 Endogenous Growth

Now consider an extension of the above problem in which we partially endogenize the rate of technological growth. In particular, let the increase in z be proportional to the population size, so that we have

$$\dot{z} = \eta L$$

for some $\eta > 0$. Call this the technology production function.

(a) What are some reasons why we might expect a technology production function of the form seen above?

(b) For a given L and z, what is the growth rate of technology g_z? In the long run, what does this imply about the ratio of L and z?

(c) Prove that long run technological growth is inevitable in this model, meaning immiseration is not possible. Find an expression for the long-run growth rate of technology z and the standard of living $\frac{Y}{L}$?

(d) As before, find the marginal value of an incremental improvement in technology z and consider its implications for the given technology production function.

3 Dynamic Stability

Consider the following basic Solow model with no population growth, no technological progress, and

$$\begin{aligned} C + I = Y &= F(K, L) \\ \dot{K} &= I - \delta K \\ I &= sY \end{aligned}$$

This exercise consists of describing the dynamic properties of the steady state of the model under different representations of the production function $F(K, L)$. To this end, find the steady state of the model, write down the solution to the dynamic equation associated with initial condition $K(0) = K_0$, and determine whether or not there is asymptotic convergence to the steady state for each of the production functions listed below.

(a) Cobb-Douglas production function

$$F(K, L) = K^{\alpha} L^{1-\alpha} \text{ with } 0 < \alpha < 1$$

For this case, also determine how the capital share α affects the speed of convergence to steady state and provide some intuition for this relationship.

(b) AK production function

$$F(K, L) = AK \text{ with } A > 0$$

(c) CES production function

$$F(K, L) = \left[\alpha K^{1-\rho} + (1-\alpha)L^{1-\rho}\right]^{\frac{1}{1-\rho}} \text{ with } \rho > 0 \text{ and } 0 < \alpha < 1$$

4 Balanced Growth Paths

Characterize the balanced growth path (BGP) for each of the following extensions of the Solow model. In particular, for each model derive the growth rates of the aggregate variables (g_Y, g_C, g_K, g_I) as functions of the parameters of the model (e.g., s, δ, n, g). In all cases, assume that $g_A = g > 0$ and $g_L = n > 0$.

In addition, in each of the following exercises, discuss how you would define steady-state variables (i.e., variables that are constant along the BGP of the model). For instance, recall that in the Solow model we define the steady state in terms of $k = K/(AL)$ that that k^* is constant in a BGP.

(a) Model with CES production function

$$\begin{aligned}
Y &= \left[\alpha K^{1-\rho} + (1-\alpha)(AL)^{1-\rho}\right]^{\frac{1}{1-\rho}} \\
Y &= C + I \\
I &= sY \\
\dot{K} &= I - \delta K
\end{aligned}$$

where $\alpha \in (0, 1)$ and $\rho > 0$.

(b) Model with increasing returns to scale in production

$$\begin{aligned}
Y &= K^{\alpha}(AL)^{\beta} \\
Y &= C + I \\
I &= sY \\
\dot{K} &= I - \delta K
\end{aligned}$$

where $\alpha + \beta > 1$.

(c) Model with production of capital. Rather than the usual law of motion for capital, we now have that capital is produced using a specific combination of current capital and new investment. In particular, the law of motion for capital is given by

$$\dot{K} = K^{\beta} I^{\gamma}$$

with $\beta + \gamma > 1$, while the rest of the model is described by the usual equations

$$\begin{aligned} Y &= K^{\alpha}(AL)^{1-\alpha} \\ Y &= C + I \\ I &= sY \end{aligned}$$