Midterm Review Solutions

1 Rise of the Machines

(a) The growth rate of capital is

$$\dot{k} = s_k y - (\delta + n)k$$

The growth rate of robots is

$$\dot{m} = s_m y - (\delta + n)m$$

The production function is

$$y = k^\alpha (1+m)^{1-\alpha}$$

(b) The growth rate of capital is

$$g_k = \frac{\dot{k}}{k} = s_k \left(\frac{m}{k}\right)^{1-\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - (\delta + n)$$

The growth rate of robots is

$$g_m = \frac{\dot{m}}{m} = s_m \left(\frac{k}{m}\right)^{\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - (\delta + n)$$

The growth rate of output is

$$\begin{aligned} \dot{y} &= \alpha \left(\frac{y}{k}\right) \dot{k} + (1-\alpha)\left(\frac{y}{1+m}\right)\dot{m} \\ \Rightarrow \quad \frac{\dot{y}}{y} &= \alpha \frac{\dot{k}}{k} + (1-\alpha)\left(\frac{m}{1+m}\right)\frac{\dot{m}}{m} \\ \Rightarrow \quad g_y &= \alpha g_k + (1-\alpha)\left(\frac{m}{1+m}\right) g_m \end{aligned}$$

(c) In steady state, k and m should have the same growth rate ($g_k = g_m$)

$$s_k \left(\frac{m}{k}\right)^{1-\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - (\delta + n) = s_m \left(\frac{k}{m}\right)^{\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - (\delta + n)$$

which implies

$$\frac{s_k}{s_m} = \frac{k}{m}$$

and hence

$$g_k = g_m = s_k^{\alpha} s_m^{1-\alpha} \left(\frac{1+m}{m}\right)^{1-\alpha} - (\delta + n)$$

There are two cases. The first is that both k and m grow without bound at rate

$$g = s_k^{\alpha} s_m^{1-\alpha} - (\delta + n)$$

However, when this figure is negative, we get no growth ($g_k = g_m = 0$) in the limit, and

$$m = \frac{1}{\left(\frac{\delta+n}{s_k^{\alpha} s_m^{1-\alpha}}\right)^{\frac{1}{1-\alpha}} - 1}$$

and

$$k = \frac{1}{\left(\frac{\delta+n}{s_k}\right)^{\frac{1}{1-\alpha}} - \frac{s_m}{s_k}}$$

(d) The marginal product of labor is

$$
\begin{aligned}
\frac{\partial Y}{\partial L} &= (1-\alpha)\left(\frac{K}{L+M}\right)^{\alpha} \\
&= (1-\alpha)\left(\frac{k}{1+m}\right)^{\alpha} \\
&= (1-\alpha)\left(\frac{k}{m}\right)^{\alpha}\left(\frac{m}{1+m}\right)^{\alpha}
\end{aligned}
$$

So in the continual growth case, we get

$$
\frac{\partial Y}{\partial L} = (1-\alpha)\left(\frac{s_k}{s_m}\right)^{\alpha}
$$

In the no growth case, we get

$$
\begin{aligned}
\frac{\partial Y}{\partial L} &= (1-\alpha)\left(\frac{s_k}{s_m}\right)^{\alpha}\left(\frac{s_k^{\alpha} s_m^{1-\alpha}}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}} \\
&= (1-\alpha)\left(\frac{s_k}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

2 Capital Utilization

(a) Here the utility function can be expressed as

$$
U = \int_0^{\infty} \log(c)\exp(-(\rho-n)t)dt
$$

Normalized production is given by

$$
y = xk^{\alpha}
$$

and the law of motion for capital is

$$
\dot{k} = i - (\delta(x)+n)k
$$

The Hamiltonian is then

$$H = u(xk^{\alpha} - i) + \mu(i - (\delta(x) + n)k)$$

(b) Optimality here implies

$$H_i = 0 \quad \Rightarrow \quad u'(c) = \mu$$

and

$$H_x = 0 \quad \Rightarrow \quad u'(c)k^{\alpha} = \mu k \delta'(x)$$

and

$$(\rho - n)\mu - \dot{\mu} = H_k \quad \Rightarrow \quad (\rho + \delta(x))\mu - \dot{\mu} = x\alpha k^{\alpha-1}u'(c)$$

Simplifying

$$\frac{\dot{c}}{c} = -\frac{\dot{\mu}}{\mu} = x\alpha k^{\alpha-1} - \rho - \delta(x)$$

and

$$\delta'(x) = k^{\alpha-1}$$

(c) In steady state

$$\rho + \delta(x) = xk^{\alpha-1} = \alpha x \delta'(x)$$

which implies

$$\frac{\rho + \delta(x)}{\delta(x)} = \alpha \frac{x\delta'(x)}{\delta(x)} = \alpha\varepsilon$$

$$\Rightarrow \quad \delta(x) = \frac{\rho}{\alpha\varepsilon - 1}$$

Then

$$k = \left[\frac{1}{\delta'(x)}\right]^{\frac{1}{1-\alpha}}$$

and

$$\begin{aligned} c &= xk^{\alpha} - \delta(x)k = k[xk^{\alpha-1} - \delta(x)] \\ &= k[x\delta'(x) - \delta(x)] = (\varepsilon - 1)\delta(x)k \end{aligned}$$

(d) We require $\alpha\varepsilon > 1$ for the existence of a steady state.

For the transition, low capital means high intensity x. What about y? We can write

$$y = x\left[\frac{1}{\delta'(x)}\right]^{\frac{\alpha}{1-\alpha}} \sim x^{1-(\varepsilon-1)\frac{\alpha}{1-\alpha}} = x^{\frac{1-\alpha\varepsilon}{1-\alpha}}$$

So then as long as a steady state exists, the level out output will be below steady state and converge monotonically.