Midterm Review

1 Rise of the Machines

Consider a variant of the Solow model in which there are potentially two types of capital: traditional capital K and robotic capital M, which is a perfect substitute for labor L. The production function is given by

$$Y = K^{\alpha}(L + M)^{1-\alpha}$$

Labor grows at constant rate n, and the laws of motion for these two types of capital are

$$\begin{aligned} \dot{K} &= s_k Y - \delta K \\ \dot{M} &= s_m Y - \delta M \end{aligned}$$

Meanwhile, consumption is $C = (1 - s_k - s_m)Y$.

(a) Reformulate the model in per capita terms, denoting variables normalized by L with lower case letters.

(b) What are the growth rates of k, m, and y in terms of the state variables k and m?

(c) Find the steady state of this mode. What are the limiting behaviors of k, m, and y? Provide any parameter conditions necessary. [**Hint**: there may be multiple cases.]

(d) What is the limiting behavior of the marginal product of labor in this model?

2 Capital Utilization

Now consider a Ramsey-type model where the producer can use capital more or less intensely. When used more intensely, it generates more output but depreciates faster. In particular, for a chosen level of intensity x, capital depreciates at rate $\delta(x)$. Thus in the aggregate we have

$$Y = xK^{\alpha}L^{1-\alpha}$$
$$\dot{K} = I - \delta(x)K$$

where $\delta(0) = 0$, $\delta'(0) = 0$, $\delta''(\cdot) > 0$, and $\lim_{x\to\infty} \delta(x) = \infty$.

(a) Formulate the Hamiltonian for the social planner's problem in per capita terms and describe what the respective choice and state variables are. [Note that x can be chosen freely at each instant.]

(b) Derive conditions characterizing the optimal path of investment i, capital k, utilization intensity x.

(c) Supposing that the depreciation function is specifically $\delta(x) = x^{\varepsilon}$. Find the steady state values for $\delta(x)$, k, and c.

(d) What conditions are required for the existence of a steady state? Under these conditions, supposing we start from a capital level below steady state, what do the transition paths for x, k, and y look like?