Economics 101

Lecture 1 - Preferences and Utility

1 Preliminaries

Aside from a willingness to learn and work hard, the only prerequisite for this course is mathematical knowledge. You should know basic set and function notation and be comfortable with calculus. Ensuring this is the case early on will save you a lot of grief in the future. Below is a list of some of the notation we'll be using.

1.1 Mathematical Notation

Set Inclusion: $y \in X, y \notin X$

$$X = \{1, 2\} \text{ implies } 1 \in X \text{ and } 2 \in X$$

Subsets: $X \subseteq Y$, $X \subset Y$

$$\{1, 2\} \subset \{1, 2, 3\}$$

Implication: $A \Rightarrow B$ reads "A implies B", meaning "If A, then B". $A \Leftrightarrow B$ means $A \Rightarrow B$ and $B \Rightarrow A$, reads "A if and only if B".

Quantifiers:

$$\begin{aligned} x \geq 0 \quad \forall \quad x \in Z \quad &\text{(universal)} \\ \exists \quad x \in Z \quad \text{s.t.} \quad x \geq 0 \quad &\text{(existential)} \end{aligned}$$

These can be related

$$\sim [\exists \quad x \in Z \quad \text{s.t.} \quad x \geq 0] \Leftrightarrow [x < 0 \quad \forall \quad x \in Z]$$

Common Sets:

$$\mathbb{R} = (-\infty, \infty) \qquad \mathbb{R}_+ = [0, \infty) \qquad \mathbb{R}_{++} = (0, \infty) \qquad \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

Calculus: Partial derivative $\frac{\partial}{\partial x}$. Total derivative $\frac{d}{dx}$.

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

Vectors: $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Dot product $(\cdot)$:

$$x \cdot y = x_1 y_1 + x_2 y_2$$

2 Preferences

Preferences are the foundational concept of microeconomics. They are descriptions of how people rank various outcomes relative to one another. Given preferences, we should be able to make statements about how people will act in particular situations. Of course, preferences cannot be directly observed, but by observing people's decisions, we should be able infer things about their preferences.

Consider a single economic agent. Let X be the set of possible outcomes. This is an exhaustive list of outcomes that could occur within the setting at hand. It might be various bundles of goods (e.g. one apple, two oranges), amounts of money, amounts of leisure time, etc.

Preference Relation: an ordering on X. For any two elements of X, it tells you which one is preferred by the agent.

Preferences relations can be thought of as subsets of X^2. Let $R \subset X^2$ be a preference relation. We say that x is preferred to y if $(x, y) \in R$. Another way to express this is $x \succeq y$.

Weak Preference: $x \succeq y$ means the agent weakly prefers x to y, that is he either strictly prefers x to y or is indifferent between the two.

Strict Preference: $x \succ y$ means the agent strictly prefers x to y. Mathematically, we define this as $x \succeq y$ and $y \not\succeq x$.

Indifference: $x \sim y$ means the agent is indifferent between x and y. This is defined as $x \succeq y$ and $y \succeq x$.

Notice that the fundamental object here is R which defines weak preferences. Strict preference and indifference are derived from these.

2.1 Properties

The following is a list of common properties of preference relations.

Completeness: $\forall x, y$, either $x \succeq y$ or $y \succeq x$, or both.

Reflexivity: $\forall x$, $x \succeq x$ ($\Rightarrow x \sim x$ and $x \not\succ x$)

Transitivity: $\forall x, y, z$, $[x \succeq y$ and $y \succeq z] \Rightarrow x \succeq z$

Anti-Symmetry (Strictness): $\forall x, y$ $(x \neq y)$, $x \succeq y \Rightarrow y \not\succeq x$

The first three properties are quite reasonable to expect in the wild, and will be assumed throughout the class. The final one will in general not hold.

Example 1. Let X be a set composed of different models of car

$$X = \{\text{Elantra}, \text{Prius}, \text{Tesla}, \text{Aztec}\}$$

Suppose the preference relation R ranks them as

R
Tesla
Prius - Elantra
Aztec

Notice that in order to write preferences in table form, as above, we need them to be complete and transitive. In terms of set notation, we have

$$\begin{aligned} R = \{&(T,T),(T,P),(T,E),(T,A),(P,P),\\ &(P,E),(P,A),(E,E),(E,P),(E,A),(A,A)\} \end{aligned}$$

Regarding $\succeq$, we can say by construction

$$x \succeq y \quad \forall\ (x, y) \in R$$

We can also say the following about strict preferences: $T \succ P$, $T \succ E$, $T \succ A$, $P \succ A$, $E \succ A$. And finally for indifference, $P \sim E$.

2.2 Derived Sets

Given a preference relation, we can define the following sets

Weakly Preferred Set:

$$WP(x) = \{y \in X \big| y \succeq x\}$$

Weakly Dominated Set:

$$WD(x) = \{y \in X \big| x \succeq y\}$$

Indifference Set:

$$I(x) = \{y \in X \big| y \sim x\} = WP(x) \cap WD(x)$$

In addition, we can define strictly preferred sets and strictly dominated sets. Notice that we can also relate these sets to the properties of preference relations listed above. For instance reflexivity implies $x \in WP(x)$, while completeness implies that for any $x, y \in X$, $y \in WP(x) \cup WD(x)$.

2.3 Continuity

Much like functions, we can speak of preferences being continuous. We will see in a little bit that this is intricately related to expressing preferences using utility functions.

Continuity: A preference relation R ($\succeq$) is continuous if, given a sequence $\{x^n\}_{n=1}^{\infty}$ with $x^n \to x$ and $x^n \succeq y$ $\forall n$, then $x \succeq y$.

In words, this means that if you weakly prefer points that are very close to x to y, then you should also weakly prefer x to y. For finite X, this is automatically true. Things get more interesting when we move to the real numbers.

Example 2. Let $X = [0, 1]$. Suppose R is defined so that

$$x \succeq y \Leftrightarrow x \geq y$$

This satisfies all of the properties listed above. In addition, it is continuous, since the inequality is preserved in limits.

Example 3. Again assume $X = [0, 1]$. This time Let R be defined by

$$x \succeq y \Leftrightarrow [x \geq y \text{ and } x \neq 1/2]$$
$$x \succeq 1/2 \ \forall \ x \in X$$

In this case, assume that $x = 1/2$ and $y = 1/4$. Consider the sequence $x^n = 1/2 + 1/2n$. This satisfies the premise that $x^n \to x$ and $x^n \succeq y$ for all n. However, $x \not\succeq y$, so these preferences are not continuous.

3 Utility Functions

We still have the set of possible outcomes X, but now we have a function $u : X \to \mathbb{R}$ that ranks the choices in X. A utility function works much like a preferences relation. However, we shall see that they are not equivalent.

Representation: A utility function u represents R ($\succeq$) if

$$u(x) \geq u(y) \Leftrightarrow x \succeq y \qquad \forall \ x, y \in X$$

Any utility function can be represented by a preference relation. Simply define the preference relation according to the above formula. Furthermore, said relation will be complete and transitive. On the other hand, not every preference relation can be represented by a utility function. The following theorem sheds some light on the situation.

Theorem 1 (Debreu)**.** *A preference relation admits a utility function representation if it is complete, transitive, and continuous.*

Notice that there are discontinuous preference relations that admit a utility representation, such as Example 3 above. In other words, this is only a sufficient condition, not a necessary condition.

Example 4. Let $X = \{1, 2, 3\}$. Define a preference relation by

$$R = \{(1, 2), (2, 3), (3, 1)\}$$

In other words $1 \succ 2$, $2 \succ 3$, and $3 \succ 1$. Let u be a utility function that represents R, with $u(1) = a$, $u(2) = b$, and $u(3) = c$. From the above, we can conclude $a > b$, $b > c$, and $c > a$, which implies $a > a$. This is a contradiction, so it must be that there is no such u.

Example 5. Let $X = [0,1]^2$. Define what are called Lexicographic preferences according to

$$x \succeq y \Leftrightarrow x_1 \geq y_1 \text{ or } [x_1 = y_1 \text{ and } x_2 \geq y_2] \quad \forall\, x, y \in X$$

They are so called because the ordering over points in X is that of words in a dictionary: sort by the first number and use the second number in case of a tie.

These preferences can be considered pathological. There is no amount of good 1 that the agent is willing to give up for any amount of good 2. We can verify that these are complete, transitive, and strict. However, they are not continuous. To see this, consider the sequence

$$x^n = (1/2 + 1/2n, 1/2) \to (1/2, 1/2) = x$$

Let $y = (1/2, 3/4)$. For all n, we have $x^n \succeq y$. However, $x \not\succeq y$, so these preferences cannot be continuous.

3.1 Properties

Utility functions are invariant under increasing transformations. Suppose we define a new utility function $w(x) = g(u(x))$, and g is increasing. Then

$$w(x) \geq w(y) \quad \Leftrightarrow \quad g(u(x)) \geq g(u(y)) \quad \Leftrightarrow \quad u(x) \geq u(y)$$

So w and u represent the same preferences. What this means is that, as a matter of philosophical interpretation, we treat utility functions as ordinal rather than cardinal. That is, only the relative ranking of two choices is important, not the magnitude of their utility difference. An implication of this stance is that interpersonal utility comparisons are meaningless, since each individual's utility can be scaled arbitrarily.

The following is a list of commonly discussed properties of utility functions. They apply for general X, however, for most of the class we will restrict attention to $\mathbb{R}^N_+$ or subsets thereof.

Increasingness: For all $x, y \in X$ with $x > y$

$$u(x) \geq u(y)$$

Concavity: For all $x, y \in X$ and $\theta \in [0,1]$ where $z^\theta = \theta x + (1-\theta)y$

$$u(z^\theta) \geq \theta u(x) + (1-\theta)u(y)$$

Quasi-concavity: For all $x, y \in X$ and $\theta \in [0,1]$ where $z^\theta = \theta x + (1-\theta)y$

$$u(z^\theta) \geq \min\{u(x), u(y)\}$$

Each of these also has a strict counterpart, where the final inequality is strict rather than weak. Notice that concavity implies quasi-concavity since

$$u(z^\theta) \geq \theta u(x) + (1-\theta)u(y) \geq \min\{u(x), u(y)\}$$

As with preferences, we can define sets based on the relative ranking of various elements of X. Along those lines, we have

Upper Contour Set:

$$UC(x|u) = \left\{y \in X \middle| u(y) \geq u(x)\right\}$$

Lower Contour Set:

$$UC(x|u) = \left\{y \in X \middle| u(y) \leq u(x)\right\}$$

Indifference Set:

$$UC(x|u) = \left\{y \in X \middle| u(y) = u(x)\right\}$$

We can relate to the sets to properties of utility functions. For instance, if utility is strictly increasing, the indifference sets are curves, rather than sets of positive volume. In such a case, these curves also cannot cross. See if you can depict a proof of this.

Fact. A utility function u is quasi-concave if and only if $UC(x|u)$ is a convex set for all x. That is, for all $y, w \in UC(x|u)$ and all $\theta \in [0,1]$

$$\theta y + (1-\theta)w \in UC(x|u)$$

3.2 Common Functional Forms

All of the following commonly used utility functions are defined over $X = \mathbb{R}^2_+$. Try to imagine what the indifference curves will look like for all of these.

Cobb-Douglas: This utility function is extremely common. It is given according to

$$u(x, y) = x^\alpha y^{1-\alpha}$$

where $\alpha \in (0, 1)$. We can also, without loss of generality, express it under the transformation

$$w(x, y) = \log u(x, y) = \alpha \log(x) + (1 - \alpha) \log(y)$$

This has the advantage of being easier to work with.

Quasi-linear: Utility function given according to

$$u(x, y) = x + v(y)$$

for some function v. We can think of x as "cash" to be spent on lots of other goods that we don't model.

Satiated: Given according to

$$u(x, y) = -(x - x_0)^2 - (y - y_0)^2$$

We call (x_0, y_0) the agent's "bliss point," as it gives higher utility than any other point. Notice that this is concave, but not increasing.

Leontieff: Given by

$$u(x, y) = \min\{ax, by\}$$

for some $a > 0$ and $b > 0$. Here, x and y are called perfect compliments, since they only yield utility when they are consumed in the proper proportion. One example is left-shoes and right-shoes. A less contrived one would be coffee and cream, where a/b represents a person's ideal ratio.

3.3 Debreu's Theorem Revisited

We can argue the correctness of Debreu's Theorem visually using indifference curves. Here, we assume that utility is increasing. Consider some point a in X. By assumption, there must be a point b on the identity line with $a \succ b$ and a point c on the identity line with $c \succ a$.

By continuity, there must be some point d on the identity line such that $a \sim d$. By increasingness, the identity line must be split contiguously between points which are preferred to a and points that a is preferred to. On the boundary we must find d.

Since d is on the identity line, it must be of the form $d = (x, x)$. So let our proposed utility function assign utility x to point a. This function must represent the preferences in question. If there were points w and z such that $w \succ z$ and $u(w) < u(z)$, this would imply that two indifference curves had crossed, which we showed earlier is impossible.

3.4 The Derivative of Utility

For much of the class, we'll be assuming $X = \mathbb{R}_+^N$. Each dimension x_i of an element of the choice set $x \in X$ represents a quantity of a good, and the entire vector is called a bundle of goods.

Marginal utility = derivative of utility. It's not deep.

When we start studying optimization in exchange economies, the derivative of utility will start being important. However, the derivative of utility still suffers from the lack of a canonical scale. To get around this, we can consider the ratio between the derivative of utility with respect to two goods. We'll use the notation $u_i(x) = \frac{\partial u(x)}{\partial x_i}$ to denote derivatives.

Marginal Rate of Substitution (MRS): The ratio of marginal utilities between two goods. For $i, j \in \{1, \ldots, N\}$

$$MRS_{ij}(x|u) = -\frac{u_i(x)}{u_j(x)}$$

Notice that the derivatives and the MRS itself are functions of x. This measure has the advantage of being invariant to monotonic transformation of utility. We'll show this in a little bit.

First, let's interpret the MRS. Locally, the MRS can be viewed as the amount of good i you must give me in exchange for one unit of good j in order to keep me indifferent. Take some achievable utility level $\overline{u}$. Under reasonable assumptions, for any value of x_i, there should be an x_j that makes my utility equal to $\overline{u}$. Let this be the function $x_j(x_i|\overline{u})$. For any x_i, it satisfies

$$u(x_i, x_j(x_i|\overline{u})) = \overline{u}$$

Since the left side of the above equation is constant over x_i, its derivative should be zero

$$u_i(x) + u_j(x) \cdot \frac{\partial x_j}{\partial x_i} = 0$$

Rearranging yields

$$\frac{\partial x_j}{\partial x_i} = -\frac{u_i(x)}{u_j(x)} = MRS_{ij}(x|u)$$

Now, the graph of this function $x_j(x_i|\overline{u})$ is actually just an indifference curve. It yields the set of points that give one exactly $\overline{u}$ utility. So it turns out that the MRS at x is the slope of the indifference curve passing through x.

As mentioned earlier, the MRS is invariant to increasing transformation of utility. Let $w(x) = g(u(x))$, where g is an increasing function. Then

$$MRS_{ij}(x|w) = -\frac{w_i(x)}{w_j(x)} = -\frac{g'(u(x)) \cdot u_i(x)}{g'(u(x)) \cdot u_j(x)} = -\frac{u_i(x)}{u_j(x)} = MRS_{ij}(x|u)$$

So the MRS is indeed a robust measure of how an agent values two goods relative to one another. Strictly speaking, understanding the MRS is not essential accomplishing what we'll be doing in the coming lectures, but it may provide some intuition.

Example 6. Suppose your utility is linear

$$u(x, y) = ax + by$$

Then the MRS is simply

$$MRS(x, y) = -\frac{u_x}{u_y} = -\frac{a}{b}$$

So when utility is linear, the MRS is constant. In general, the MRS will not be constant

Example 7. Now let utility be

$$u(x, y) = \sqrt{x} + \sqrt{y}$$

The MRS is now

$$MRS(x, y) = -\frac{u_x}{u_y} = -\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}} = -\sqrt{\frac{y}{x}}$$

So as your consumption of x approaches zero, the amount of y needed to compensate for one unit of lost x goes to infinity. This could be the case, for instance, if x and y were food and water.

3.5 Concavity

You should know by now that if u is increasing, then $u_i \geq 0$ for all i, and similarly when strict. But what about the second derivative? Here we extend notation to include $u_{ii} = \frac{\partial^2 u}{\partial x_i^2}$.

Proposition 1. *If u is concave, then $u_{ii} \leq 0$.*

Proof. We'll prove this in one dimension. Consider two arbitrary points $x, y \in X$. By concavity

$$\begin{aligned} & u(\theta x + (1-\theta)y) \geq \theta u(x) + (1-\theta)u(y) \\ \Rightarrow\ & u(y + \theta(x-y)) - u(y) \geq \theta(u(x) - u(y)) \\ \Rightarrow\ & \left[\frac{u(y + \theta(x-y)) - u(y)}{\theta(x-y)}\right](x-y) \geq u(x) - u(y) \end{aligned}$$

Taking the limit as $\theta \to 0$ yields

$$u'(y)(x-y) \geq u(x) - u(y)$$

Noting that we could have done the same things with the roles of x and y reversed, we also have

$$u'(x)(y-x) \geq u(y) - u(x)$$

Combining these two inequalities yields

$$\begin{aligned} & u'(y)(x-y) \geq u(x) - u(y) \geq u'(x)(x-y) \\ \Rightarrow\ & \left[u'(x) - u'(y)\right](x-y) \leq 0 \\ \Rightarrow\ & \frac{u'(x) - u'(y)}{x-y} \leq 0 \end{aligned}$$

Taking the limit as $x \to y$ yields $u'' \leq 0$.

$\square$